

Pure Connection Formulation, Twistors and the Chase for a Twistor Action for General Relativity

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Abstract

This paper establishes the relation between traditional results from (euclidean) twistor theory and chiral formulations of General Relativity, especially the pure connection formulation. Starting from a $SU(2)$ -connection only we show how to construct natural complex data on twistor space, mainly an almost Hermitian structure and a connection on some complex line bundle. Only when this almost Hermitian structure is integrable is the connection related to an anti-self-dual-Einstein metric and makes contact with the usual results. This leads to a new proof of the non-linear-graviton theorem. Finally we discuss what new strategies this "connection approach" to twistors suggests for constructing a twistor action for gravity. In appendix we also review all known chiral Lagrangians for GR.

Introduction

It is well known that gravity can be given "chiral formulations" ¹, ie formulations where the full local isometry group $SO(4, \mathbb{C}) = SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$ (we consider complexified gravity for generality) loses its central role for one of the "chiral" (left or right) subgroup $SL(2, \mathbb{C})$. This shift in the local symmetries corresponds to a shift in the hierarchy of fields: in "chiral formulations" of GR the role of the metric is usually played down for other alternative variables with natural $SL(2, \mathbb{C})$ internal symmetries. The metric then typically only appears as a derived object and its associated local isometry group $SO(4, \mathbb{C})$ comes as an "auxiliary symmetry" that was somewhat hidden in the first place.

It's probably safe to say that the interest of the physics community for such reformulations started with Ashtekar "new" variables [4] and the appealing form of the related diffeomorphism constraints. In subsequent works [2], [5] it was understood that the (in fact ten year older!) Plebanski's action [1] gave a covariant description of Ashtekar variables. In

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¹For the most striking ones see [1], [2], [3]. See also Appendix A for a review of the Chiral Lagrangians for gravity

Plebanski pioneering work the metric disappears completely for $SL(2, \mathbb{C})$ -valued fields. In both points of view, canonical and covariant, $SL(2, \mathbb{C})$ -connections play a crucial role.

That $SL(2, \mathbb{C})$ -connections appear is no surprise: already in the more traditional metric perspective, the Levi-Cevita connection comes as an $SO(4, \mathbb{C})$ -connection. The decomposition of Lie group $SO(4, \mathbb{C}) = SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$ then corresponds to a splitting of the Levi-Cevita connection into Left(or self-dual) and Right(or anti-self-dual) $SL(2, \mathbb{C})$ -connection, which are in some sense the most natural "chiral" objects one can construct from the metric. What the "chiral formulations" of GR do is essentially to reverse this construction: using $SL(2, \mathbb{C})$ fields (eg connections) as a building blocks for the metric. This culminates in the so called "pure connection of GR" pursued in [6] and finally achieved in [3] where the only field that appear in the Lagrangian is an $SL(2, \mathbb{C})$ -connection.

On the other hand it is not always realised that twistor theory, at least in its original Penrose's program [7] directed towards gravity, has a very nice interplay with these reformulations and is in fact part of "chiral formulations" of GR in a broad sense. This is more clearly seen by taking a closer look at the main result of twistor theory on the gravity side, the "non-linear graviton theorem" [8], [9].

The "non-linear graviton theorem" takes as a starting point a 8 dimensional real manifold (the twistor space) equipped with an almost complex structure. This reduces the group of local symmetry to $SL(4, \mathbb{C})$ which is also the 4d (complex)conformal group $SO(6, \mathbb{C}) \simeq SL(4, \mathbb{C})$ and indeed the first half of the non-linear graviton theorem asserts that, under some generic conditions, integrability of this almost complex structure is equivalent to a 4d complexified conformal anti-self-dual space-time (ie such that self-dual part of Weyl curvature vanishes). That this theorem only describes anti-self-dual space-time clearly points in the direction of the intrinsic chirality of twistor theory but there is more.

The second half of the theorem requires additional data on Twistor space in the form of a certain complex 1-form up to scale, usually denoted as τ , ie essentially a 4d real distribution at every point (the kernel of τ). This 1-form is taken to have certain compatibility with the almost complex structure such that its kernel is in turn almost complex and identifies with \mathbb{C}^2 . The restriction of the symmetry group $SL(4, \mathbb{C})$ to this distribution thus brings us down to the "Chiral" group $SL(2, \mathbb{C})$: In fact such a 1-form most naturally is associated with a "Chiral" $SL(2, \mathbb{C})$ -connection on space-time (This is especially clear in the Euclidean context and we will come back to this in what follows). As connections are not conformally invariant, it fixes a scale in the conformal space-time. The second part of the Non-Linear-Graviton Theorem then essentially asserts that this scale is such that the resulting metric is anti-self-dual Einstein if one is given a "good enough" (holomorphic) 1-form.

The usual approach to twistor theory generally emphasises the metric aspect of the theorem and somewhat overlook the fact that this 1-form, which crucially fixes the right scaling to give Einstein equations, is directly related to a $SL(2, \mathbb{C})$ -chiral connection thus putting twistor theory in the general framework of "Chiral formulations of gravity". In section 3 we will review the basics of the curved Twistor construction with an emphasis on the relation between the chiral connection and the $\mathcal{O}(2)$ -valued 1-form on Twistor space τ .

It is in fact well known to specialists that *there is* an interplay between, for example, Plebanski formulation of GR and Twistor theory as can be seen from the introduction of Twistor variables in some recent spin-foam models [10],[11] or in the conjoint use of Plebanski action and Twistor theory [12] to investigate the structure of MHV gravity amplitudes. It is however still possible that not all consequences have been drawn from this overlapping.

Now, one of the "most radical" Chiral formulation of GR is the pure connection formulation where only a $SL(2, \mathbb{C})$ -connection is considered as fundamental field, the metric being a derived object. Einstein equations then take the form of certain second order field equations on the connection.

In this paper we would like to emphasise the change of perspective on twistor theory that this extreme chiral reformulation of gravity suggests: We already stated that the equivalent on twistor space of this Chiral connection on space-time is a complex 1-form, τ . In usual twistor theory this form is just taken to be some additional data that only complements the almost complex structure which is considered as fundamental. However the pure connection formulation of GR suggests that it is the 1-form τ (loosely related to the chiral connection) that should be taken as fundamental with the almost complex structure (related the conformal structure) arising as a derived objects.

We demonstrate in section 4 that, at least in the Euclidean signature context, it is a valuable point of view and that it allows to reproduce nicely the results from the non-linear-graviton theorem while putting twistor theory firmly into the "Chiral formulations" framework of gravity:

For a Riemannian manifold M (ie equipped with a metric of Euclidean signature) the associated twistor space $\mathbb{T}(M)$ is simply taken to be the 2-spinor bundle. Then an $SU(2)$ -connection, $A^{A'}_{B'}$, allows to define the 1-form on $\mathbb{T}(M)$

$$\tau = \pi_{A'} \left(d\pi^{A'} + A^{A'}_{B'} \pi^{B'} \right)$$

which relates to the preceding discussion.

We first show that an $SU(2)$ -connection is enough to construct a Hermitian structure on $\mathbb{PT}(M)$, thus making contact with usual Euclidean twistor theory:

Proposition 0.1. Almost Hermitian structure on $\mathbb{PT}(M)$

If A is a definite connection (see below for a clarification of this notion) then $\mathbb{PT}(M)$ can be given an almost Hermitian structure on $\mathbb{PT}(M)$, ie a compatible triplet $(\mathcal{J}_A, \omega_A, g_A)$ of almost complex structure, 2-form, and a Riemannian metric.

In general this triplet is neither Hermitian (J_A is not integrable) nor almost Kähler (Ω_A is non degenerate but generically not closed). In fact integrability of J_A is equivalent to the statement that A is the self-dual connection of a self-dual Einstein metric with non zero cosmological constant. The metric on twistor space can be made Kähler if and only if A is the self-dual connection of a self-dual Einstein metric with positive cosmological constant (ie if the connection is of "positive sign").

Further more, the integrability condition is equivalent to $\tau \wedge d\tau \wedge d\tau = 0$.

The main difference with the traditional results from [13] is that integrability is not only related to the anti-self-duality but is irremediably linked to Einstein equations. This is because in the construction described in [13] one is only interested in a conformal class of metric while here the use of the connection automatically fixes the "right scaling" that gives Einstein equations.

The fact that the connection needs to be "definite" refers to a natural non-degeneracy condition. Such connection can be assigned a sign. This terminology first appeared in [14] and we will review it in section 2. The possibility of associating a symplectic structure on $\mathbb{PT}(M)$ with a definite $SU(2)$ -connection on M was already pointed out in [14]. However,

only in the integrable case does the symplectic structure described in this reference coincides with our ω_A . $SL(2, \mathbb{C})$ -connections which are the self-dual connection of a self-dual Einstein metric with non zero cosmological constant were called “perfect” in [15] and are the one such that their curvature verify $F^i \wedge F^j \propto \delta^{ij}$. This well known (see eg [16]) description of Einstein anti-self-dual metric in terms of connection will also be reviewed in section 2.

On the other hand starting with a certain 6d manifold \mathcal{PT} , the projective twistor space, together with a 1-form valued in a certain line bundle τ , we have a variant of the non linear graviton theorem:

Proposition 0.2. Pure connection Non-Linear Graviton Theorem

If τ is a definite 1-form then \mathcal{PT} can be given an almost complex structure J_τ .

Together with some compatible conjugation operation on \mathcal{PT} this is enough to give \mathcal{PT} the structure of a fibre bundle over a 4d manifold $M: \mathbb{CP}^1 \hookrightarrow \mathcal{PT} \rightarrow M$.

Integrability of J_τ is then equivalent to the possibility of writing τ as

$$\tau = \pi_{A'} \left(d\pi^{A'} + A^{A'}{}_{B'} \pi^{B'} \right)$$

with A the self-dual connection of a Einstein anti-Self-Dual metric on M with non zero cosmological constant.

What's more the integrability condition reads $\tau \wedge d\tau \wedge d\tau = 0$.

Parts of this last proposition were already known and developed in [17],[18] and [19] as part of a strategy to obtain twistor actions for conformal gravity, anti-self-dual gravity and gravity (this last action being still missing). We however give here a new proof that emphasises the role of the connection as a fundamental object and we hope that by framing them in the general perspective of Chiral approach to gravity they will appear in a new light, ie as more than just clever trick to construct Twistor action. In particular we hope to make it clear that one can effectively think of the (euclidean)non-linear-graviton theorem as a far reaching generalisation of the description of Einstein Anti-Self-Dual metric in terms of connections.

Our long term view in developing what could be called a ”connection approach” to twistor theory, with the 1-form τ being the main field instead of the almost complex structure, was that it could open new strategies to construct twistor action for gravity. However one faces difficulties the we could not overcome. We briefly explain in section 4 our work in this direction and why it does not seem to offer a way to a twistor action for gravity.

This paper is organised as follows: In section 1 we review chiral formulations of gravity with an emphasis on the general geometric setting underlying any formulation of this type rather than on a particular Lagrangian. We especially stress how to write equations for self-dual gravity in this framework and review the pure connection field equations for Einstein anti-self-dual metric and Einstein metric. This will serve as a model for our ”connection version” of the non-linear-graviton theorem. For completeness we gathered in appendix different Lagrangians that belong to the chiral gravity type, some of which might be unfamiliar to the reader. They will be also useful when considering the problem of a twistor action for gravity.

In section 2 we review Twistor theory in Euclidean signature (cf [20], [13]) but from an unusual connection perspective, ie where we take a $\mathfrak{su}(2)$ -connection to be the main field instead of a metric. From this data only we show how to construct very natural structures on twistor space, namely the 1-form τ , some associated connection on $\mathcal{O}(n)$ bundle and the triplet (\mathcal{J}, ω, g) of compatible almost complex structure, 2-form and Euclidean metric on twistor space of Prop 0.1. We also review from [14] some symplectic structure that is naturally constructed from the connection. Finally we investigate the condition for integrability of the almost complex structure or for which the triplet (\mathcal{J}, ω, g) is Kähler. These cases turn out to be given by the self-dual-gravity equations and therefore make contact with the usual Kähler structure on twistor space constructed from an instanton (ie an anti-self-dual Einstein metric).

We then state and give a new proof for the non linear graviton theorem from a pure connection point of view (cf Prop 0.2).

Finally in section 3 we explain how ideas from the previous sections suggests new ansatz for constructing twistor action for gravity. However this section will stay inconclusive and we only see ideas described there as few more elements on the chase (cf [17],[21], [19] and [22]) for this elusive (if existing) action.

1 Chiral Formulations of Gravity

1.1 Geometrical Foundations

Chiral formulations of gravity exploits the fact that Einstein equations can be stated using only “one half ” of the decomposition $SO(4, \mathbb{C}) = SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$. We here briefly review why this is possible.

The whole discussion in this section could be treated in complexified terms but for clarity and coherence with the other sections we will restrict to the real form $SO(4, \mathbb{R}) = SU(2) \times SU(2)$, ie Euclidean signature.

Chiral decomposition of the curvature tensor

We now consider a Riemannian manifold (M, g) . We note $\{e^I\}_{I \in 0..4}$ an orthonormal frame, it is defined up to $SO(4)$ transformations. In order to see that Einstein equations can be stated using only one half of the decomposition $SO(4) = SU(2) \times SU(2)$, the quickest way is to split the Riemann curvature tensor into self-dual/anti-self-dual pieces. This is classically done in spinor notation (see eg [23]) or more directly as in [13]. We here make a presentation along the line of the second reference with an emphasises on the necessity of using a torsion-free connection in order for chiral formulations of gravity to be possible.

By using the metric, 2-forms can be identified with skew-adjoint transformations of Λ^1 and thus with $\mathfrak{so}(4)$:

$$\mathfrak{b} \in \mathfrak{so}(4) \quad \simeq \quad b_{IJ} \frac{e^I \wedge e^J}{2} \in \Lambda^2 \quad \simeq \quad b \in \text{End}(\Lambda^1) : a_I e^I \rightarrow b_I^J a_J e^I \quad (1)$$

Where $b_{IJ} = b_I^K g_{KJ}$. The split $\mathfrak{so}(4) \simeq \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ then corresponds to the decomposition of 2-forms into self-dual and anti-self-dual 2-forms, $\Lambda^2 = \Lambda_+^2 \oplus \Lambda_-^2$.

Consider a connection ∇ on the tangent bundle compatible with the metric, this is a $SO(4)$ -connection (Note that, at this stage, we do not assume that the torsion of this connection vanishes). It splits into two $SU(2)$ -connections D and \tilde{D} ,

$$\nabla = D + \tilde{D}.$$

They naturally act as connections on the bundle of self-dual 2-forms and anti-self-dual 2-forms respectively.

Now, the curvatures of D and \tilde{D} are a $\mathfrak{su}(2)$ -valued 2-forms,

$$D^2 \in \Lambda^2(\mathfrak{su}(2)) \simeq \Lambda^2(\Lambda_+^2), \quad \tilde{D}^2 \in \Lambda^2(\mathfrak{su}(2)) \simeq \Lambda^2(\Lambda_-^2),$$

and it follows from the decomposition, $\Lambda^2 = \Lambda_+^2 \oplus \Lambda_-^2$, that we can write them as bloc matrices:

$$D^2 = \begin{pmatrix} F & G \end{pmatrix}, \quad \tilde{D}^2 = \begin{pmatrix} \tilde{F} & \tilde{G} \end{pmatrix}$$

where $F \in \text{End}(\Lambda_+^2)$, $G \in \text{Hom}(\Lambda_+^2, \Lambda_-^2)$, $\tilde{F} \in \text{End}(\Lambda_-^2)$, $\tilde{G} \in \text{Hom}(\Lambda_-^2, \Lambda_+^2)$.

Finally, the curvature of ∇ , $\nabla^2 \in \Lambda^2(\mathfrak{so}(4)) \simeq \text{End}(\Lambda^2)$, can be written as a bloc matrix:

$$\nabla^2 = \begin{pmatrix} F & G \\ \tilde{G} & \tilde{F} \end{pmatrix}.$$

Einstein equations for the metric compatible connection ∇ then read $G = 0$, $\tilde{G} = 0$ and $\frac{\text{tr}F + \text{tr}\tilde{F}}{2} = \text{cst} = \Lambda$. In Appendix B we prove, using coordinates, that this is indeed equivalent to the usual $R_{\mu\nu} = \Lambda g_{\mu\nu}$.

It is also convenient to introduce the self-dual and anti-self-dual Weyl tensor:

$$W_+ = F - \frac{1}{3}\text{tr}F \mathbb{I}, \quad W_- = \tilde{F} - \frac{1}{3}\text{tr}\tilde{F} \mathbb{I}.$$

Without any further assumptions this is as far as we can get: we need both $G = 0$ and $\tilde{G} = 0$ to state Einstein equations. However, in the special case of the Levi-Cevita connection, ie if one assumes that the connection is torsion free, we get a simpler picture. $\nabla^2: \Lambda^2 \rightarrow \Lambda^2$ is then the usual Riemann curvature tensor and has some further symmetries. Namely, it is then symmetric, ie $\tilde{G} = G^t$, $W^+ = (W^+)^t$, $W^- = (W^-)^t$, and what is more, $\text{tr}F = \text{tr}\tilde{F}$. Using coordinates, one can indeed immediately see that this is equivalent to the usual first Bianchi identity, this is done in Appendix B.

We then get the celebrated decomposition of the Riemann tensor into irreducible components:

$$\nabla^2 = \text{tr}F \mathbb{I}_{\Lambda^2} + \begin{pmatrix} 0 & G \\ G^t & 0 \end{pmatrix} + \begin{pmatrix} W_+ & 0 \\ 0 & W_- \end{pmatrix}. \quad (2)$$

From which it stems that,

$$g \text{ is Einstein if and only if } G = 0 \quad (3)$$

and then the scalar curvature is $4\Lambda = 4\text{tr}F$.

In particular one sees from $D^2 = F + G$ that Einstein equations can be stated in term of D only: The metric is Einstein if and only if D is a self-dual gauge connection, ie if D^2 is a self-dual $\mathfrak{su}(2)$ -valued 2-form.

From this presentation it should be clear that this general phenomenon stems from the internal symmetries of the Riemann tensor, related to torsion freeness, and note from a particular choice of signature.

Urbantke metric

We explained how Einstein equations can be stated in an essentially chiral way, ie in terms of $\mathfrak{su}(2)$ -valued fields. This general principle underlies any chiral formulation of gravity. However this was still very classical in spirit as we considered the metric as the fundamental field. We now describe an essential observation due to Urbantke [24] that allows to obtain a metric as a derived object from chiral (ie $su(2)$ -valued) fields.

Suppose that we have a $\mathfrak{su}(2)$ -valued 2-forms B , using a basis of $\mathfrak{su}(2)$, $\{\tau_i\}_{i \in 1,2,3}$, this gives us a triplet of real 2-form $\{B^i\}_{i \in 1,2,3}$, such that $B = B^i \tau_i$

Now, given such a triplet of 2-forms $\{B^i\}_{i \in 1,2,3}$, there is a unique conformal structure that makes the triplet (B^1, B^2, B^3) self-dual. We will refer to this conformal structure as the Urbantke metric associated with B and write it as $\tilde{g}_{(B)}$. There is even a way to make this conformal structure explicit through Urbantke formula [24],

$$\text{Urbanke metric:} \quad \tilde{g}_{(B)\mu\nu} = -\frac{1}{12} \tilde{\epsilon}^{\alpha\beta\gamma\delta} \epsilon_{ijk} B^i_{\mu\alpha} B^j_{\nu\beta} B^k_{\gamma\delta}. \quad (4)$$

Obviously, if the B 's do not span a 3 dimensional vector-space this cannot hold. In fact the "metric" (4) will then be degenerated in the sense that it will not be invertible. A more precise statement, again from [24], is the following: given the triplet of two forms (B^1, B^2, B^3) , defines the conformal "internal metric" $\tilde{X}^{ij} = B^i \wedge B^j / d^4x$ then Urbantke metric $\tilde{g}_{(B)}$ is invertible if and only if \tilde{X} is. When Urbantke metric is invertible \tilde{X} is just the metric on the space of self-dual 2-forms given by wedge product.

As we started with a triplet $\{B^i\}_{i \in 1,2,3}$ of *real* 2-forms, the associated Urbantke metric (4) is also real. On the other hand, its signature is undefined: self-dual 2-forms in Lorentzian signature are complex so this signature is excluded but without further restriction it can still be either Euclidean or Kleinian. The signature of the internal metric \tilde{X} however is enough information to fix this ambiguity: for an Euclidean conformal metric \tilde{g} the metric \tilde{X} on self-dual 2-forms given by wedge product is Euclidean while for a Kleinian signature it would be Lorentzian.

Thus if we start with a triplet $\{B^i\}_{i \in 1..3}$ of real 2-form such that the internal metric $\tilde{X}^{ij} = B^i \wedge B^j / d^4x$ is definite, we are then assured that the associated Urbantke metric, $\tilde{g}_{(B)}$ is non degenerate (invertible) and of Euclidean signature. This suggests to introduce the following definition:

Definition 1. Definite Triplet of 2-forms

A triplet (B^1, B^2, B^3) of real 2-form is called definite if the conformal metric constructed from the wedge product $\tilde{X}^{ij} = B^i \wedge B^j / d^4x$ is definite.

As we just explained this is a useful definition because of the following:

Proposition 1.1. Urbantke metric

The Urbantke metric (4) associated with a definite triplet of 2-forms is non degenerate and of Euclidean signature.

In this section we made two distinct but complementary observations, first Einstein equations can be stated in a chiral way (cf equation (3)) ie using $\mathfrak{su}(2)$ -valued fields, second a (definite) $\mathfrak{su}(2)$ -valued 2-form is enough to define a metric. Lagrangians that realise "Chiral formulations" of GR all rely on some mixture of these two facts each with its own flavour and fields.

For the interested reader, we gathered in Appendix A some explicit Lagrangians and further references.

However, at least in the first parts of this paper, we won't be interested by a particular action but rather by how the general framework that we just describe intersect with twistor theory. Our main guide will be the description of anti-self-dual Einstein metric in terms of connections.

Before we come to this it is useful to introduce two new tensors.

Two useful tensors: the sigma 2-forms

We already made the remark that a metric allows to identify the Lie algebra $\mathfrak{so}(4)$ with the space of 2-forms Λ^2 , we denote by

$$\sigma: \mathfrak{so}(4) = \mathfrak{su}(2) \oplus \mathfrak{su}(2) \rightarrow \Lambda^2 = \Lambda_+^2 \oplus \Lambda_-^2$$

this isomorphism.

We choose a basis $\{\tau^i, \tilde{\tau}^i\}_{i \in 1,2,3}$ of $\mathfrak{so}(4) = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ adapted to the decomposition and such that $[\tau^i, \tau^j] = \epsilon^{ijk} \tau^k$, $[\tilde{\tau}^i, \tilde{\tau}^j] = \epsilon^{ijk} \tilde{\tau}^k$, $[\tau^i, \tilde{\tau}^j] = 0$. Then one can define the sigma 2-forms:

$$\frac{1}{2} \Sigma^i = \sigma(\tau^i), \quad \frac{1}{2} \tilde{\Sigma}^i = \sigma(\tilde{\tau}^i).$$

Thus $\{\Sigma^i\}_{i \in 1,2,3}$ (resp $\{\tilde{\Sigma}^i\}_{i \in 1,2,3}$) form a basis of self-dual 2-forms Λ_+^2 (resp anti-self-dual 2-forms Λ_-^2). This basis is also defined (up to $SU(2)$ transformations) by the orthogonality relations

$$\Sigma^i \wedge \Sigma^j = \tilde{\Sigma}^i \wedge \tilde{\Sigma}^j = 2\delta^{ij} Vol_g, \quad \Sigma^i \wedge \tilde{\Sigma}^j = 0.$$

The awkward factor of one half in the definition is there for it to fit with the definition in terms of a tetrad that frequently appears in the literature:

$$\left\{ \Sigma^i = -e^0 \wedge e^i - \frac{\epsilon^{ijk}}{2} e^j \wedge e^k \right\}_{i \in 1,2,3}, \quad \left\{ \tilde{\Sigma}^i = e^0 \wedge e^i - \frac{\epsilon^{ijk}}{2} e^j \wedge e^k \right\}_{i \in 1,2,3}. \quad (5)$$

The sigma 2-forms are naturally $\mathfrak{su}(2)^*$ -valued 2-forms or, using the Killing metric on $\mathfrak{su}(2)$, $\mathfrak{su}(2)$ -valued 2-forms:

$$\Sigma = \Sigma^i \tau^i \in \Lambda_+^2(\mathfrak{su}(2)), \quad \tilde{\Sigma} = \tilde{\Sigma}^i \tilde{\tau}^i \in \Lambda_-^2(\mathfrak{su}(2)).$$

Importantly they are compatible with the connections $D = d + A$, $\tilde{D} = d + \tilde{A}$, in the following sense:

$$d_A(\Sigma) = d\Sigma + [A, \Sigma] = 0, \quad d_{\tilde{A}}(\tilde{\Sigma}) = d\tilde{\Sigma} + [\tilde{A}, \tilde{\Sigma}] = 0. \quad (6)$$

See Appendix B for a direct proof in coordinates. This compatibility relations are important as they can be used as alternative definition for the chiral connection D and \tilde{D} , ie the two equations (6) have only one solution.

Finally we can write the Einstein equations in terms of those 2-forms. If $D = d + A$ is the "left" or "self-dual" connection then we can rewrite the first half of the bloc decomposition (2) as

$$D^2 = F^i \tau^i = \left(F^{ij} \Sigma^j + G^{ij} \tilde{\Sigma}^j \right) \tau^i \quad (7)$$

Then, as we already discussed, the self-dual part of Weyl curvature is $F^{ij} - \frac{1}{3} \text{tr} F \delta^{ij}$, the scalar curvature is $4\Lambda = 4 \text{tr} F$ and

$$g \text{ is Einstein if and only if } D^2 = M^{ij} \Sigma^j \tau^i. \quad (8)$$

1.2 Definite Connections and Gravity

We review here how to write equations for Einstein-anti-self-dual metric in terms of connections. This is a well known construction (cf [16]) and we here use the terminology of [14],[15]. We also briefly recall how to write equations for full Einstein gravity in terms of connections from [3],[25].

We now take $A^i \tau^i$ to be the potential in a trivialisation of a $SU(2)$ -connection $D = d + A$ and $D^2 = F^i \tau^i$ its curvature.

Definite Connections

We mainly consider definite connections, ie connections such that the curvature 2-form is a definite triplet:

Definition 2. Definite Connections

A $SU(2)$ -connection $D = d + A^i \tau^i$, is called definite if the conformal metric, $\tilde{X}^{ij} = F^i \wedge F^j / d^4 x$, constructed from its curvature, $D^2 = F^i \tau^i$, is definite.

For any $SU(2)$ -connection with potential $A^i \tau^i$, there is a unique conformal class of metric $\tilde{g}_{(F)}$ such that the curvature $F^i \tau^i$ is self-dual. The definiteness of the connection then ensures that this conformal metric is invertible and of Euclidean signature (cf def 1 and prop 1.1). Thus definite connections are associated with a "good" metric.

A definite connections also defines a notion of orientation. It is done by restricting to volume form μ_+ such that $\tilde{X}^{ij} = F^i \wedge F^j / \mu_+$ is *positive* definite. In the following whenever there is a need for an orientation, we will always take this one.

We can also assign a *sign* to a connection as follows: We consider co-frame $\{e^I\}_{I \in 0..3}$, orthonormal with respect to the Urbantke metric and oriented with the convention that we just described. They are defined up to Lorentz transformations and rescaling by a *positive* function. From this tetrad we can construct a basis of self-dual 2-form $\{\Sigma^i\}_{i \in 1,2,3}$ through the relation (5). Again $\{\Sigma^i\}_{i \in 1,2,3}$ is defined up to $SU(2)$ transformations and rescalings by *positive* functions. By construction, the curvature $D^2 = F^i \tau^i$ is self-dual for the associated Urbantke metric and we can thus write

$$D^2 = F^i \tau^i = M^{ij} \Sigma^j \tau^i.$$

The sign of the connection is then defined as $\sigma = \text{sign}(\det(M))$. Note that this notion of sign makes sense as a result of $\{F^i\}_{i \in 1,2,3}$ being defined up to $SU(2)$ transformations and $\{\Sigma^i\}_{i \in 1,2,3}$ being defined up to $SU(2)$ transformations and positive rescaling.

We now have two $SU(2)$ transformations independently acting on $\{F^i\}_{i \in 1,2,3}$ and $\{\Sigma^i\}_{i \in 1,2,3}$, the first as a result of changing the trivialisation of the $SU(2)$ principal bundle of whom $D = d + A$ is a connection, the second as a result of changing the trivialisation of the bundle of self-dual 2-forms associated with the Urbantke metric. Those two bundles can be identified (at least locally) by requiring M^{ij} to be a definite symmetric matrix. Finally we also have two scaling transformations, one acting on \tilde{X} and the other one on Σ , we identify them by requiring that $F^i \wedge F^j = \tilde{X}^{ij} \frac{1}{3} \Sigma^k \wedge \Sigma^k$.

In what follows these identifications will always be assumed unless we explicitly specify otherwise.

As a result of \tilde{X}^{ij} being definite we can make sense of its square root. In fact there is a slight ambiguity in this definition: we fix it by requiring $\sqrt{\tilde{X}}$ to be positive definite, ie we take the positive square root.

With these choices of square root and identifications, we have

$$F^i = \sigma \sqrt{\tilde{X}}^{ij} \Sigma^j \quad \Leftrightarrow \quad \Sigma^i = \sigma \sqrt{\tilde{X}}^{-1ij} F^j, \quad \forall i \in 1, 2, 3.$$

Self-dual gravity and Perfect Connections

A metric is said to be "anti-self-dual" if the self-dual part of its Weyl curvature vanishes ie, if $W_+ = 0$ in (2). As Weyl curvature is conformally invariant, this is a property of the conformal class of the metric rather than from the metric itself.

A metric is Einstein-anti-self-dual if it is Einstein and anti-self-dual, ie if $W_+ = 0$, $G = 0$ in (2). Alternatively, using (7), if

$$F^i = \frac{\Lambda}{3} \Sigma^i, \quad \forall i \in 1, 2, 3. \quad (9)$$

(then 4Λ is the scalar curvature). Note in particular that for $\Lambda \neq 0$, $F^i \wedge F^j / d^4x \propto \delta^{ij}$.

This motivates the following definition,

Definition 3. Perfect Connections

A definite connection is perfect if $F^i \wedge F^j = \delta^{ij} \frac{F^k \wedge F^k}{3}$.

The relevance of this definition comes from the following:

Proposition 1.2.

The Urbantke conformal metric associated with a perfect connection is anti-self-dual. What's more the representative with volume form $\frac{3}{2\Lambda^2} F^k \wedge F^k$ is anti-self-dual-Einstein with cosmological constant $\sigma|\Lambda|$, where σ is the sign of the connection.

Proof.

Consider the Urbantke metric with volume form $\mu = \frac{3}{2\Lambda^2} F^k \wedge F^k$. It is associated with a orthonormal basis of 2-form $\{\Sigma^i\}_{i \in 1,2,3}$ as in (5). By construction, they are such that $\Sigma^i \wedge \Sigma^j = 2\delta^{ij} \mu$. Together with our identification of the scaling transformations, $F^i \wedge F^j = \tilde{X}^{ij} \frac{1}{3} \Sigma^k \wedge \Sigma^k$, it gives

$$F^i \wedge F^j = 2\tilde{X}^{ij} \mu.$$

Now by hypotheses,

$$F^i \wedge F^j = \frac{\delta^{ij}}{3} F^k \wedge F^k = 2\delta^{ij} \frac{\Lambda^2}{9} \mu,$$

from which we read $X^{ij} = \frac{\Lambda^2}{9}\delta^{ij}$ and

$$F^i = \sigma\sqrt{X}^{ij}\Sigma^j = \sigma\frac{|\Lambda|}{3}\Sigma^i. \quad (10)$$

From this last relation we see that Bianchi identity, $d_A F = 0$, is now equivalent to $d_A \Sigma = d\Sigma + [A, \Sigma] = 0$ which is the defining equation (6) of the self-dual connection. It follows that $D = d + A$ is the self-dual connection of the Urbantke metric with volume form $\mu = \frac{3}{2\Lambda^2}F^k \wedge F^k$. With this observation (10) are just the field equations for Einstein anti-self-dual gravity (9) with cosmological constant $\sigma|\Lambda|$. \square

Pure connection formulation of Einstein equations

At this point it is hard to resist writing down the pure connection formulation of Einstein equations. However we won't use them in the rest of this paper.

Consider a definite $SU(2)$ -connection $D = d + A$ with curvature $F = F^i \tau^i$. As already explained, it is associated with an orientation, a sign σ and conformal class of metric $\tilde{g}_{(F)}$. We again denote $F^i \wedge F^j = \tilde{X}^{ij} d^4x$ and define the following volume form,

$$\frac{1}{2\Lambda^2} \left(\text{Tr} \sqrt{F \wedge F} \right)^2 := \frac{1}{2\Lambda^2} \left(\text{Tr} \sqrt{\tilde{X}} \right)^2 d^4x. \quad (11)$$

This is a well defined expression as a result of the following facts: the definiteness of the connection together with the orientation make \tilde{X}^{ij} positive definite and thus we can take its square root, what's more $\left(\text{Tr} \sqrt{\tilde{X}} \right)^2$ being homogeneous degree one in \tilde{X} the overall expression does not depends on the representative of the density \tilde{X} .

However there are signs ambiguity in this choice of square root. They amount to the choice of signature of the conformal metric \sqrt{X}^{ij} . We will always take this choice of square root such that $\det(\sqrt{X}) > 0$, then the only signatures that remains are $(+, +, +)$ and $(+, -, -)$. We thus need to make a choice for our definition of square root once and for all: either we stick with the “definite square root” or with the “indefinite square root”.

Definition 4. Einstein Connections

If A^i is a definite connection, define X^{ij} by the relation

$$F^i \wedge F^j = 2X^{ij} \frac{1}{2\Lambda^2} \left(\text{Tr} \sqrt{F \wedge F} \right)^2. \quad (12)$$

Then we will call it Einstein if

$$d_A \left(\sqrt{X}^{-1ij} F^j \right) = 0 \quad (13)$$

.

Again, the two square roots in this definition need to be taken with the same convention, ie such that the resulting matrices have the same signature: either $(+, +, +)$ (“definite square root”) or indefinite $(+, -, -)$ (“indefinite square root”). Note that for perfect connections, $X^{ij} = \delta^{ij} \frac{\Lambda^2}{9}$, as a result of which perfect connections are special case of Einstein connections with the “definite square root” convention (note that perfect connections

are *not* Einstein connections for the “indefinite square root” as $d_A \left(\sqrt{X}^{-1ij} F^j \right) \neq 0$ for $\sqrt{X} = \text{diag}(1, -1, -1)$.

The Definition 4 is motivated by the following,

Proposition 1.3. Krasnov [3]

For an Einstein connection, the Urbantke metric with volume form $\frac{1}{2\Lambda^2} \left(\text{Tr} \sqrt{F \wedge F} \right)^2$ is Einstein with cosmological constant $|\Lambda| \text{Sign} \left(\sigma \text{Tr} \sqrt{F \wedge F} \right)$. What’s more such a connection coincides with the self-dual Levi-Cevita connection of the metric.

Proof.

The metric in Urbantke conformal class with volume form $\nu = \frac{1}{2\Lambda^2} \left(\text{Tr} \sqrt{F \wedge F} \right)^2$ is associated with an orthonormal basis of self-dual 2-form $\{\Sigma^i\}_{i \in 1,2,3}$, $\Sigma^i \wedge \Sigma^j = 2\delta^{ij}\nu$. It is defined up to $SU(2)$ transformation. By definition, $\{F^i\}_{i \in 1,2,3}$ is a basis of self-dual 2-forms for Urbantke metric and $F^i = M^{ij}\Sigma^j \forall i \in \{1, 2, 3\}$.

As was already pointed out, *a priori* F and Σ are valued in two different associated $SU(2)$ bundle: $D = d + A$ is a $SU(2)$ connection on a $SU(2)$ principal bundle P and the curvature naturally takes value in the adjoint bundle $P \times_{SU(2)} \mathfrak{su}(2)$, on the other hand $\{\Sigma^i\}_{i \in 1,2,3}$ is a trivialisation of the bundle of self-dual 2-forms associated with the Urbantke metric.

We now come again to the subtle question of identifying the two: this can be done (at least locally) by requiring M^{ij} to be a symmetric matrix. Once this is done, however there is still the possibility of acting with the diagonal transformation $(\Sigma^1, \Sigma^2, \Sigma^3) \rightarrow (\Sigma^1, -\Sigma^2, -\Sigma^3)$ and we thus have two possible identifications. We call them the “definite identification” and the “indefinite identification” depending whether or not the resulting matrix M^{ij} is definite or not.

As a rule, we now take the identification corresponding to the square root that we chose, ie if one chooses the “definite square root”, we take the “definite identification”; on the other hand, if one takes the “indefinite square root” one should use the “indefinite identification”.

Finally, just as in the case of perfect connections, we identify rescaling of \tilde{X} and rescaling of Σ by imposing that $F^i \wedge F^j = \tilde{X}^{ij} \frac{1}{3} \Sigma^k \wedge \Sigma^k$. Together with the choice of volume form, $\Sigma^i \wedge g\Sigma^j = 2\delta^{ij}\nu$, this completely fixes all the scaling freedom: $F^i \wedge F^j = 2X^{ij}\nu$. Note that this gives the same result as in definition 4.

As a consequence of these different choices we have

$$F^i = \sigma \sqrt{X}^{ij} \Sigma^j \quad \Leftrightarrow \quad \Sigma^i = \sigma \sqrt{X}^{-1ij} F^j.$$

The field equations (13) now read $d_A \Sigma = 0$ which are just the the defining equations (6) of the self-dual connection. It follows that $D = d + A$ is the self-dual connection of the Urbantke metric with volume form ν . Having this in mind, $F^i = \sigma \sqrt{X}^{ij} \Sigma^j$, are Einstein equations (8) with cosmological constant $\sigma \text{Tr} \left(\sqrt{X} \right)$. Finally, from (12), one gets $|\text{Tr} \left(\sqrt{X} \right)| = |\Lambda|$. \square

Note that one of the weakness of this formulation is that a particular choices of square root (ie “definite” or “indefinite”) can only describe a particular subspace of Einstein metric, those such that the self-dual Weyl curvature F^{ij} is respectively definite or indefinite.

Interestingly, the integral of the volume form $\frac{1}{2\Lambda^2} \left(\text{Tr} \sqrt{F \wedge F} \right)^2$ also gives the correct variational principle for Einstein connections. This is the *pure connection action* for GR

[3]. It can be obtained by integrating fields successively from the Plebanski action, see also Appendix A.

2 Euclidean Twistor Theory Revisited: a Connection Point of View

We now review the Euclidean version of the twistor construction but from an unusual "connection point of view".

We take "space-time" as a $SU(2)$ -principal bundle $SU(2) \hookrightarrow P \rightarrow M$ over a 4d manifold M equipped with a $SU(2)$ -connection $D = d + A$. We will describe this connection by its potential in a trivialisation, $A = A^i \tau^i$.

The associated "twistor space" $\mathbb{T}(M)$ is simply the 2-spinor bundle over M : $\mathbb{C}^2 \hookrightarrow \mathbb{T}(M) \rightarrow M$. We will use adapted local coordinates $(x^\mu, \pi_{A'})$ to describe this bundle. As always, we will raise and lower spinor indices with the anti-symmetric tensor $\epsilon_{A'B'}$ (Here this is simply the metric volume form preserved by the $SU(2)$ action). Having $SU(2)$ structure group, the \mathbb{C}^2 fibers of this bundle come equipped with a hermitian metric that is commonly represented by an anti-linear, anti-involutive map,

$$\hat{\cdot} : \begin{cases} \mathbb{C}^2 \rightarrow \mathbb{C}^2 \\ \pi_{A'} \rightarrow \hat{\pi}_{A'} \end{cases}$$

such that

$$\alpha, \beta \in \mathbb{C}^2, \quad \langle \alpha, \beta \rangle := \beta_{A'} \hat{\alpha}^{A'}.$$

Making use of the fundamental representation of $SU(2)$, the $SU(2)$ -connection $D = d + A$ naturally acts as a connection on twistor space :
if

$$s \begin{cases} M \rightarrow \mathbb{T}(M) \\ x \rightarrow \pi_{A'}(x) \end{cases}$$

is a section of $\mathbb{T}(M)$ then its covariant derivative with respect to A is

$$\nabla \pi_{A'} = d\pi_{A'} - A^{B'}_{A'} \pi_{B'}, \quad A^{A'}_{B'} \in \mathfrak{su}(2). \quad (14)$$

Now we can also re-interpret this last equality in terms of forms: We define the 1-forms $D\pi_{A'} \in \Omega^1(\mathbb{T}(M))$ on the full space of the bundle $\mathbb{T}(M)$ as

$$D\pi_{A'} = d\pi_{A'} - A^{B'}_{A'} \pi_{B'} \in \Omega^1(\mathbb{T}(M)). \quad (15)$$

These are in fact the coordinates of a projection operator, the projection operator on the vertical tangent space to $\mathbb{T}(M)$:

$$Proj = D\pi_{A'} \otimes \frac{\partial}{\partial \pi_{A'}} \in \text{End}(T\mathbb{T}(M)). \quad (16)$$

The kernel of this operator is the *horizontal distribution* associated with the connection $D = d + A$. Thus (14), (15) corresponds to the usual dual points of view on connections:

either as a differential operators acting on section or as a horizontal distribution on the total space of the bundle.

The associated "projective Twistor space" $\mathbb{PT}(M)$ is the projectivised version of $\mathbb{T}(M)$, with Fiber isomorphic to \mathbb{CP}^1 : $\mathbb{CP}^1 \hookrightarrow \mathbb{PT}(M) \rightarrow M$. We will most frequently use homogeneous coordinates $(x^\mu, [\pi_{A'}])$ to describe this bundle. The main advantage with this notation is that section in $\mathcal{O}(n, m)$ -bundle over \mathbb{CP}^1 (and by extension over $\mathbb{PT}(M)$) are equivalent to functions $f(x, \pi_{A'})$ with homogeneity n in $\pi_{A'}$ and m in $\hat{\pi}_{A'}$.

Similarly k-forms on $\mathbb{PT}(M)$ with values in $\mathcal{O}(n, m)$ are uniquely represented by k-forms on $\mathbb{PT}(M)$ with homogeneity n in $\pi_{A'}$, m in $\hat{\pi}_{A'}$ which vanishes on $E = \pi_{A'} \frac{\partial}{\partial \pi_{A'}}$, $\bar{E} = \hat{\pi}_{A'} \frac{\partial}{\partial \hat{\pi}_{A'}}$. These two vectors are the "Euler vectors", they generate the vertical tangent space of the complex line bundle $\mathbb{C} \hookrightarrow \mathbb{T}(M) \rightarrow \mathbb{PT}(M)$.

$$\alpha' \in \Omega^k(\mathbb{PT}, \mathcal{O}(n, m)) \Leftrightarrow \alpha \in \Omega^k(\mathbb{T}) \quad \text{st} \quad E \lrcorner \alpha = 0, \quad \bar{E} \lrcorner \alpha = 0, \quad \mathcal{L}_E \alpha = n\alpha, \quad \mathcal{L}_{\bar{E}} \alpha = m\alpha.$$

For example

$$\tau := \pi_{A'} D\pi^{A'}$$

represents a $\mathcal{O}(2)$ -valued 1-form on $\mathbb{PT}(M)$ but $\hat{\pi}_{A'} D\pi^{A'}$ does not represent a well defined object on $\mathbb{PT}(M)$ as it does not vanish on $\text{Span}(E, \bar{E})$.

We can use this fact to define a connection on the $\mathcal{O}(n, m)$ bundles. For suppose $f(x, \pi_{A'})$ represents a section of the $\mathcal{O}(n, m)$ bundle, $\mathcal{L}_E f = nf$, $\mathcal{L}_{\bar{E}} f = mf$. Then we can define its covariant derivative as

$$d_{(n, m)} f := df + n \frac{\hat{\pi}_{A'} D\pi^{A'}}{\pi \cdot \hat{\pi}} f - m \frac{\pi_{A'} D\hat{\pi}^{A'}}{\pi \cdot \hat{\pi}} f$$

It is a simple exercise to verify that $E \lrcorner d_{(n, m)} f = 0$, $\bar{E} \lrcorner d_{(n, m)} f = 0$, $\mathcal{L}_E d_{(n, m)} f = n d_{(n, m)} f$, $\mathcal{L}_{\bar{E}} d_{(n, m)} f = m d_{(n, m)} f$ and thus that $d_{(n, m)} f$ indeed represents a $\mathcal{O}(n, m)$ -valued 1-form on $\mathbb{PT}(M)$.

This connection also preserves the following Hermitian metric on the $\mathcal{O}(n, m)$ -bundles:

$$\alpha, \beta \in \mathcal{O}(n, m), \quad \langle \alpha, \beta \rangle = \bar{\alpha} \beta (\pi \cdot \hat{\pi})^{-n-m}. \quad (17)$$

It is indeed a simple calculation to check that $d_{(n, n)} (\pi \cdot \hat{\pi})^n = 0$. In particular, when restricted to each \mathbb{CP}^1 this connection is the natural Chern-connection on $\mathcal{O}(n)$ bundle induced by the Kähler structure.

This connection on $\mathcal{O}(n, m)$ bundle over $\mathbb{PT}(M)$ extends to a connection on $\mathcal{O}(n, m)$ -valued k-forms in the usual way. It is for example instructive to check that,

$$d_{(2)} \tau = F^{A'}_{B'} \pi_{A'} \pi^{B'}.$$

We thus see that the $SU(2)$ -connection that we started with induces two natural geometric objects on $\mathbb{PT}(M)$: a $\mathcal{O}(2)$ -valued 2-forms $\tau = \pi_{A'} D\pi^{A'}$ and a covariant derivative $d_{(n, m)}$ on the $\mathcal{O}(n, m)$ -bundle over $\mathbb{PT}(M)$.

2.1 Symplectic and almost Hermitian structure on $\mathbb{PT}(M)$ from a definite connection

We now restrict ourselves to the case of definite connections (2), ie the case where $\tilde{X}^{ij} = F^i \wedge F^j / d^4 x$ is a definite 3x3 conformal metric. This is in fact equivalent to the requirement

that no real 3-vector $\{v^i\}_{i \in 1,2,3}$ is such that $v^i F^i$ is a simple 2-form:

$$A \text{ is a definite connection} \quad \Leftrightarrow \quad \forall v^i \in \mathbb{R}^3, \quad v^i F^i \wedge v^j F^j = v^i v^j \tilde{X}^{ij} d^4x \neq 0.$$

A definite connection on $\mathbb{PT}(M)$ naturally gives a symplectic structure:

Proposition 2.1. Symplectic structure on $\mathbb{PT}(M)$ (Fine and Panov [14])

If A is a definite connection then $\omega_s = (n - m)^{-1} (d_{(n,m)})^2$, $n \neq m$, is a symplectic structure on $\mathbb{PT}(M)$.

Proof.

As $d_{n,m}$ is a connection on a line bundle, its curvature 2-form

$$\omega_s = (n + m)^{-1} (d_{(n,m)})^2 = (n - m)^{-1} d \left(\frac{n \hat{\pi}_{A'} D\pi^{A'} - m \pi_{A'} D\hat{\pi}^{A'}}{\pi \cdot \hat{\pi}} \right)$$

is automatically closed. A direct computation shows that,

$$\omega_s = \frac{\pi_{A'} D\pi^{A'} \wedge \hat{\pi}_{B'} D\hat{\pi}^{B'}}{(\pi \cdot \hat{\pi})^2} - F^{A'B'} \frac{\pi_{A'} \hat{\pi}_{B'}}{\pi \cdot \hat{\pi}} \quad (18)$$

and therefore ω_s is independent of n and m . From this last expression one also sees that non degeneracy is equivalent to the definiteness of the connection. \square

We also have an almost Hermitian structure obtained by a modification of the classical one described in [13],[20]:

Proposition 2.2. Almost Hermitian structure on $\mathbb{PT}(M)$

If A is a definite connection then $\mathbb{PT}(M)$ can be given an almost Hermitian structure on $\mathbb{PT}(M)$, ie a compatible triplet $(\mathcal{J}_A, \omega_A, g_A)$ of almost complex structure, 2-form, and a Riemannian metric. In general this triplet is neither Hermitian (\mathcal{J}_A is not integrable) nor almost Kähler (ω_A is non degenerate but generically not closed).

Proof.

We first describe how to construct the almost complex structure \mathcal{J}_A on $\mathbb{PT}(M)$ from a definite connection: Because the connection is definite, one can make sense of the square root (we take the positive square root) and inverse of X . Defines $\Sigma^i = X^{-\frac{1}{2}ij} F^j$. By construction $\Sigma^i \wedge \Sigma^j \propto \delta^{ij}$. It implies that $\Sigma_\pi = \Sigma^{A'B'} \pi_{A'} \pi_{B'}$ is simple, $\Sigma_\pi \wedge \Sigma_\pi = 0$. We now define the almost complex structure by the requirement that $\Omega_A = \tau \wedge \Sigma_\pi$ be a $(3,0)$ -form. It makes sense as its kernel, $\{X \text{ st } X \lrcorner \Omega_A = 0\}$, is 3 dimensional and thus can be identified with the $(0,1)$ -distribution:

$$X \in T^{0,1}\mathbb{PT}(M) \quad \Leftrightarrow \quad X \lrcorner \tau \wedge \Sigma_\pi = 0. \quad (19)$$

This construction has a simple metric interpretation: We already explained how to construct a conformal, non degenerate, Euclidean metric from a definite connection. We will note $e^{AA'}$ the associated null tetrad. It is then easy to see that the construction leading to Σ^i is in fact just an alternative way of constructing $\Sigma^{A'B'} = \frac{1}{2} e^{A'C} \wedge e_C^{B'}$ (or equivalently (5), see Appendix C for our conventions on spinors).

One now comes to the compatible metric on $\mathbb{PT}(M)$. From the definite connection we have a conformal metric. One fixes the scaling freedom by requiring the volume form to

be $\frac{3}{2\Lambda^2}F^k \wedge F^k$. We will note $e^{AA'}$ the associated null tetrad. This gives a metric on the horizontal tangent space (as defined by A), on the other hand the vertical tangent space comes equipped with a metric and altogether this gives the following metric on $\mathbb{P}\mathbb{T}(M)$ ²:

$$g_A = 4R^2 \frac{\pi_{A'} D\pi^{A'} \odot \hat{\pi}_{B'} D\hat{\pi}^{B'}}{2(\pi \cdot \hat{\pi})^2} + \frac{1}{2} e^{AA'} \odot e_{AA'} = 4R^2 \frac{\pi_{A'} D\pi^{A'} \odot \hat{\pi}_{B'} D\hat{\pi}^{B'}}{2(\pi \cdot \hat{\pi})^2} - \frac{e^{AB'} \pi_{B'} \odot e_A{}^{C'} \hat{\pi}_{C'}}{\pi \cdot \hat{\pi}}. \quad (20)$$

We leave R , the radius of the fibers, as a parameter but we will see that the Kähler condition will relate it uniquely with Λ .

From this, one readily sees that the 2-form,

$$\omega_A = 4iR^2 \frac{\pi_{A'} D\pi^{A'} \wedge \hat{\pi}_{B'} D\hat{\pi}^{B'}}{2(\pi \cdot \hat{\pi})^2} - i \frac{e^{AB'} \pi_{B'} \wedge e_A{}^{C'} \hat{\pi}_{C'}}{\pi \cdot \hat{\pi}} = 2iR^2 \left(\frac{\pi_{A'} D\pi^{A'} \wedge \hat{\pi}_{B'} D\hat{\pi}^{B'}}{(\pi \cdot \hat{\pi})^2} - \frac{1}{R^2} \frac{\Sigma^{B'C'} \pi_{B'} \hat{\pi}_{C'}}{\pi \cdot \hat{\pi}} \right) \quad (21)$$

and the almost complex structure \mathcal{J}_A , given by the $(3,0)$ -form

$$\Omega_A^{3,0} = \tau \wedge \Sigma^{A'B'} \pi_{A'} \pi_{B'} = \pi_{A'} D\pi^{A'} \wedge e^{0A'} \pi_{A'} \wedge e^{1B'} \pi_{B'}$$

are compatible. This is clear as (19),(20),(21) are already in the canonical form

$$\Omega_A^{3,0} = dz^1 \wedge dz^2 \wedge dz^3, \quad g_A = h_{i\bar{j}} dz^i \odot d\bar{z}^{\bar{j}}, \quad \omega_A = i h_{i\bar{j}} dz^i \wedge d\bar{z}^{\bar{j}}.$$

□

Essentially this construction is a variation of the one in [13]. As compared to the classical construction from [13] there are however small differences:

First the conformal structure is obtained from the connection.

Second one does not use the notion of horizontality associated with the (Levi-Cevita connection of the) conformal structure but the one given by *our original* $SU(2)$ -connection. In general those two connections differ. The special case where they coincide in fact corresponds to the Einstein case, ie the base metric is Einstein.

For clarity, we expand a little bit on this last point even though this is more related to the pure connection formulation of Einstein equations (that we reviewed in (1.3)) and somewhat lies out of the main line of development: Suppose that, A , the $SU(2)$ -connection that we took as starting point coincide with A_g , the (Left Chiral part of the) Levi-Cevita connection, then their curvature also coincide: $F = F_g$. Now by construction Urbantke metric is such that it makes F self-dual. Therefore F_g is self-dual and this is just the chiral way of stating Einstein equations (Cf first part of section 2).

A natural question is then to ask when this almost Hermitian structure is Hermitian, ie J_A is integrable.

Proposition 2.3.

$\Leftrightarrow \mathcal{J}_A$ is integrable

$\Leftrightarrow \bar{\partial}\tau = 0$

$\Leftrightarrow d_{(n)}$ is compatible with \mathcal{J}_A , ie $(d_{(n)})^2|_{(0,2)} = 0$

²Here $A \odot B = A \otimes B + B \otimes A$

$$\begin{aligned} &\Leftrightarrow \tau \wedge d\tau \wedge d\tau = 0 \\ &\Leftrightarrow A^i \text{ is perfect : } F^i \wedge F^j \propto \delta^{ij} d^4x. \end{aligned}$$

It follows that under this conditions the $\mathcal{O}(n)$ -bundles are holomorphic with Hermitian metric (17) and $d_{(n)}$ is the associated Chern connection.

We recall that by proposition (1.2) a perfect connections is the connection of an anti-self-dual Einstein metric. In particular, under the assumption of proposition 2.3:

The Urbantke metric with volume form $\frac{3}{2\Lambda^2} F^k \wedge F^k$ is anti-self-dual Einstein.

Proof.

We now prove that each point taken separately is equivalent to perfectness of the connection.

It is easy to check that $\tau \wedge d\tau \wedge d\tau = \tau \wedge F^{A'B'} \wedge F^{C'D'} \pi_{A'} \pi_{B'} \pi_{C'} \pi_{D'}$ and thus $\tau \wedge d\tau \wedge d\tau = 0$ is directly equivalent to the perfectness of the connection.

In the previous section, we saw that $F^i = \sigma \sqrt{X}^{ij} \Sigma^j$. Thus we can write

$$F^{A'B'} = \psi^{A'B'}_{C'D'} \Sigma^{C'D'} + \lambda(x) \Sigma^{A'B'} \quad \text{with} \quad \psi^{A'B'C'D'} = \psi^{(A'B'C'D')}.$$

It was also explained in the previous section that the self-dual Einstein equations, ie perfectness of the connection, are equivalent to $\psi = 0$. Then our choice of volume form $\mu = 2\Sigma^i \wedge \Sigma^i = \frac{3}{2\Lambda^2} F^k \wedge F^k$ gives $\lambda(x) = \sigma|\Lambda|$.

A direct computation shows that

$$d\tau|_{0,2} = \Psi \pi \pi \pi \pi \frac{\Sigma \hat{\pi} \hat{\pi}}{(\pi \cdot \hat{\pi})^2}, \quad \bar{\partial}\tau = \Psi \pi \pi \pi \hat{\pi} \frac{\Sigma \pi \hat{\pi}}{(\pi \cdot \hat{\pi})^2}, \quad (d_{(n)})^2|_{(0,2)} = \Psi \pi \pi \pi \hat{\pi} \frac{\Sigma \hat{\pi} \hat{\pi}}{(\pi \cdot \hat{\pi})^2}.$$

Therefore $\tau|_{0,2} = 0$, $\bar{\partial}\tau = 0$ and $(d_{(n)})^2|_{(0,2)} = 0$ are separately equivalent to $\psi = 0$, ie to the perfectness of the connection.

Finally, all is left to show is that integrability of J_A is equivalent to the perfectness of the connection. However integrability is equivalent to having both $\tau|_{0,2} = 0$ and $d(e^{AA'} \pi_{A'})|_{0,2}$ and thus imply perfectness of the connection. On the other hand, if the connection is perfect then $d(e^{BB'} \pi_{B'})|_{0,2} = (d_A e^{BB'})|_{0,2} \pi_{B'} + e^{BB'} \wedge D\pi_{B'}|_{0,2} = (d_A e^{BB'})|_{0,2} \pi_{B'}$ holds identically as a result of A being the self-dual connection associated with the tetrad. \square

As already emphasised in the introduction, the main difference with the traditional results from [13] is that integrability is not only related to the anti-self-duality but is irremediably linked to Einstein equations. This is because in the construction described in [13] one is only interested in a conformal class of metric while here the use of the connection automatically fixes the "right scaling" that gives Einstein equations.

It is also natural to ask under which condition the almost hermitian structure is almost Kähler and Kähler. In fact those two situations necessarily come together but depends on the sign of the connection:

Proposition 2.4.

g_A is Kähler

$\Leftrightarrow J_A$ is integrable, $\sigma = 1$ and $R^2 = \frac{3}{|\Lambda|}$

$\Leftrightarrow \omega_A$ is closed (and thus symplectic)

$\Leftrightarrow \omega_A = 2iR^2 \omega_s$

Proof.

$\omega_A = 2iR^2 \omega_s$ is easily shown to be equivalent to $F^i = \frac{1}{R^2} \Sigma^i$. This is only possible if the connection is perfect with positive sign. The same is true for the closeness of ω_A , ie direct computations shows the equivalence of the last two point with $F^i = \frac{1}{R^2} \Sigma^i$.

Now, from proposition 2.3 perfectness of the connection is equivalent to integrability. Incidentally one sees from $F^i = \frac{1}{R^2} \Sigma^i$ that the metric associated with the connection is self-dual Einstein with cosmological constant $\Lambda = \frac{3}{R^2}$. \square

When the connection is perfect, the Hermitian structure that we described restrict to the usual Hermitian structure on Twistor space constructed from an Instanton. The discussion on the sign of the connection parallel the well known fact that this Hermitian structure can be made Kähler only if the cosmological constant is positive.

2.2 The Mason-Wolf action for self-dual gravity

In [21], L.Mason and M.Wolf described a twistor action for self-dual gravity. It is an action for an $\mathcal{O}(2)$ -valued 1-form τ and a $\mathcal{O}(-6)$ -valued 1-form b on some 6d real manifold, the “projective twistor space”. It essentially used a new version of the non linear graviton theorem relying on the equation $\tau \wedge d\tau \wedge d\tau = 0$. This equation was understood as a sufficient condition for the integrability of a certain almost complex structure and thus, relying on Penrose-Ward Non-Linear-Graviton theorem [8] [9], as describing some Einstein anti-self-dual space-time. The Mason-Wolf action implements this constraint with a Lagrange multiplier:

$$S[\tau, b] = \int_{\mathbb{PT}} b \wedge \tau \wedge d\tau \wedge d\tau \quad (22)$$

Even though the logic that lead to this Lagrangian was somehow different, in retrospect one sees that this Lagrangian could have been guessed from the description of self-dual gravity in terms of perfect connections. Indeed, as already explained in section 2, in terms of $SU(2)$ -connections the equations for self-dual gravity read $F^{(A'B'} \wedge F^{C'D')} = 0$ and therefore we can easily obtain an action for self-dual gravity by implementing this constraint by a Lagrange multiplier:

$$S[B, A] = \int_M B_{A'B'C'D'} F^{A'B'} \wedge F^{C'D'}, \quad (23)$$

where the B field is completely symmetric, $B^{A'B'C'D'} = B^{(A'B'C'D')}$.

Now, as discussed before the natural “Penrose transform” of a $SU(2)$ -connection is the $\mathcal{O}(2)$ -valued 1-form on $\mathbb{PT}(M)$,

$$\tau = \pi_{A'} \left(d\pi^{A'} + A^{A'}_{B'} \pi^{B'} \right).$$

We also take the Penrose transform of B to be

$$B^{A'B'C'D'} = \int_{\mathbb{CP}^1} \pi^{A'} \pi^{B'} \pi^{C'} \pi^{D'} b \wedge \tau$$

with b a $\mathcal{O}(-6)$ valued $(0,1)$ -form on $\mathbb{PT}(M)$. This is just the usual Penrose transform for massless fields, see eg [20]. We recall from the previous discussion that $\tau \wedge d\tau \wedge d\tau = \tau \wedge F^{A'B'}\pi_{A'}\pi_{B'} \wedge F^{C'D'}\pi_{C'}\pi_{D'}$.

From this one readily sees that the Mason-Wolf action (22) is the immediate generalisation of (23).

The aim of the next sub-section is to make the relation between the Mason-Wolf action and the connection description of Einstein self-dual connections even more precise by giving a new proof of the non linear graviton theorem that emphasise this relation.

Our proof will indeed make it clear that this version of the non linear graviton theorem has a strong "connection" flavour. It might therefore suggest new types of generalisation to full gravity. We discuss some of those in the next section where we will described strategies towards a twistor action for full gravity.

2.3 The Non linear graviton theorem revisited

Up to now we constructed different geometrical structure on $\mathbb{PT}(M)$ from a definite connection. In particular we saw that $\mathbb{PT}(M)$ can be given a Kähler structure when $\tau \wedge d\tau \wedge d\tau = 0$, with $\tau = \pi_{A'}(d\pi^{A'} + A^{A'}_{B'}\pi^{B'})$.

We are now interested in the reverse problem: We take " projective twistor space " \mathcal{PT} to be an oriented manifold diffeomorphic to $\mathbb{R}^4 \times S^2$ together with a 1-form $\tau \in \Omega^1_{\mathbb{C}} \otimes L$ with values in a line bundle L over \mathcal{PT} . We suppose this line bundle to be such that its restriction to each S^2 has Chern class 2. This is enough to define an almost complex structure \mathcal{J}_τ on \mathcal{PT} as we now describe.³

The almost complex structure \mathcal{J}_τ

We first introduce the 4-dimensional " horizontal distribution " $H \subset T_{\mathbb{R}}\mathcal{PT}$ defined as the kernel of τ , $H = \text{Ker}(\tau)$.

We then determine $\lambda \in C^\infty(\mathcal{PT})$ as

$$\tau \wedge \bar{\tau} \wedge (d\tau + \lambda d\bar{\tau})^2 = 0. \quad (24)$$

This is a quadratic equation for λ . We then construct a , a connection on the L bundle, defined modulo the addition of multiple of τ and $\bar{\tau}$ by requiring,

$$\bar{\tau} \wedge (\lambda d\bar{\tau} + d\tau + a \wedge \tau)^2 = 0. \quad (25)$$

This is in fact linear in a and has the right number of components to determine a modulo τ and $\bar{\tau}$. From all this we define the complex 3-form,

$$\bar{\Omega} = \bar{\tau} \wedge (\lambda d\bar{\tau} + d\tau + a \wedge \tau). \quad (26)$$

This 3-form in turn defines an almost complex structure, \mathcal{J}_τ : We just define the holomorphic tangent space to be the kernel of $\bar{\Omega}$,

$$X \in T^{1,0}\mathcal{PT} \xLeftrightarrow{\text{def}} X \lrcorner \bar{\Omega} = 0.$$

From (25) we see that the Kernel of Ω indeed is 3-dimensional as required for an almost complex structure. Note that this definition of \mathcal{J}_τ is equivalent to requiring that $\bar{\Omega}$ is $(0,3)$. In particular its complex conjugate Ω is $(3,0)$.

³We thank L.Mason for important discussions and suggestions that greatly contributed to this presentation.

Spacetime from \mathcal{J}_τ

Having constructed an almost complex structure, \mathcal{J}_τ on \mathcal{PT} we are now in a similar situation as in [17] where the almost complex structure is taken as a starting point. Following the same steps as in this reference we can construct a Euclidean space-time M from \mathcal{PT} , then \mathcal{PT} has the structure of a fiber bundle over $M : \mathbb{CP}^1 \hookrightarrow \mathcal{PT} \rightarrow M$. Twistor space \mathcal{T} is taken as the total space of a special line bundle over \mathcal{PT} .

We here recall how this works for completeness.

We first introduce a conjugation $\hat{\cdot} : \mathcal{PT} \rightarrow \mathcal{PT}$, $\hat{\cdot}^2 = 1$, that reverses \mathcal{J}_τ , i.e. $\hat{\cdot}^* \mathcal{J}_\tau = -\mathcal{J}_\tau$. We also assume that this conjugation has no fixed points. This is a common in twistor theory and will lead to a Euclidean Space-time, the other alternative (existence of fixed points for $\hat{\cdot}$) would lead to Lorentzian signature.

We now take as "complexified space-time" \mathcal{M} the moduli space of pseudo-holomorphic rational curves in \mathcal{PT} , ie the space of embedded S^2 in \mathcal{PT} in the same topological class as the S^2 factors in $\mathcal{PT} \simeq \mathbb{R}^4 \times S^2$ such that \mathcal{J}_τ leaves the tangent space invariant and thus inducing a complex structure on these embedded two-spheres. Theorems in McDuff and Salamon [26] imply that \mathcal{M} exists and is 8-dimensional if \mathcal{J}_τ is close to the standard complex structure on a neighbourhood of a line in \mathbb{CP}^3 . We assume this condition to be satisfied. This can be done by requiring that our 1-form $\tilde{\tau}$ is close to the standard holomorphic 1-form with values in $\mathcal{O}(2)$ on \mathbb{CP}^3 .

The conjugation $\hat{\cdot}$ induces a conjugation on \mathcal{M} , $\hat{\cdot} : \mathcal{PT} \rightarrow \mathcal{PT}$ and we define our *Euclidean space-time* M as the 4-dimensional fixed point set of $\hat{\cdot}$ on \mathcal{M} . There is then a natural projection $P : \mathcal{PT} \rightarrow M$ as a consequence of the fact that from our assumption that there will be a unique rational curves in \mathcal{PT} through Z and \hat{Z} . By construction our projective twistor space \mathcal{PT} now is the total space of a fibre bundle over M with fibre \mathbb{CP}^1 : $\mathbb{CP}^1 \hookrightarrow \mathcal{PT} \xrightarrow{P} M$.

We will also assume that \mathcal{J}_τ is such that the canonical bundle $\Omega^{3,0}$ has Chern class -4 on each S^2 in \mathcal{PT} . This will be the case if we construct $\tilde{\tau}$ by a small deformation of the standard holomorphic 1-form with values in $\mathcal{O}(2)$ on \mathbb{CP}^3 .

We then define the associated twistor space \mathcal{T} to be the fourth root of the canonical bundle. It is thus a complex line bundle over \mathcal{PT} , $\mathbb{C} \hookrightarrow \mathcal{T} \xrightarrow{\Pi} \mathcal{PT}$. We denote the complex line bundle $(\Omega^{3,0})^{-\frac{n}{4}}$ by $\mathcal{O}(n)$. When restricted to each \mathbb{CP}^1 fibres in \mathcal{PT} , these bundles will restrict to the usual $\mathcal{O}(n)$ holomorphic bundle on \mathbb{CP}^1 and thus the notation is coherent. We can now think of \mathcal{T} as a complex rank two vector bundle over M with structure group $SU(2)$, $\mathbb{C}^2 \hookrightarrow \mathcal{T} \xrightarrow{P'} M$.

A non linear graviton theorem

We now give a new proof of the (euclidean) non-linear-graviton theorem. As explained in introduction, the essential result of this theorem already appeared in [21] but the presentation that we make here is original.

Introduce coordinates that form a trivialisation of \mathcal{T} , $\{x^\mu, \pi_{A'}\}$. $\pi_{A'}$ with $A' \in \{0, 1\}$ are linear coordinates on the fibres of $\mathbb{C}^2 \hookrightarrow \mathcal{T} \xrightarrow{P'} M$ and x^μ are local space-time coordinates on the base.

Then,

Proposition 2.5.

- (i) J_τ is integrable
- (ii) $\Leftrightarrow \tau \wedge d\tau \wedge d\tau = 0$
- (iii) $\Leftrightarrow \tau = \gamma (\pi_{A'} d\pi^{A'} + A(x)^{A'}_{B'} \pi^{B'} \pi_{A'})$ with $\gamma \in \mathbb{C}^\infty(\mathcal{PT})$ and $A^{A'}_{B'}$ a perfect connection on M .

Proof.

We now prove $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i)$.

$(i) \Rightarrow (ii)$

By construction, $\tau \wedge (d\tau + \lambda d_{\bar{a}} \bar{\tau})^2 = \Omega \wedge (d\tau + \lambda d_{\bar{a}} \bar{\tau}) = 0$, integrability means that $\Omega \wedge d\tau = 0$ and thus $\lambda \Omega \wedge d_{\bar{a}} \bar{\tau} = 0$. It follows that either $\lambda = 0$ or $d_{\bar{a}} \bar{\tau}|_{0,2} = 0$. If $\lambda = 0$ then $\tau \wedge d\tau \wedge d\tau = 0$. Suppose $d_{\bar{a}} \bar{\tau}|_{0,2} = 0$, integrability implies that $d_{\bar{a}} \bar{\tau}|_{2,0} = 0$ and therefore $d_{\bar{a}} \bar{\tau} \in \Omega^{1,1}$. It follows that both $d_{\bar{a}} \bar{\tau} \in \Omega^{1,1}$ and $d_a \tau \in \Omega^{1,1}$. However this is in contradiction with $\tau \wedge (d\tau + \lambda d_{\bar{a}} \bar{\tau}) \in \Omega^{3,0}$.

$(ii) \Rightarrow (iii)$

If $\tau \wedge d\tau \wedge d\tau = 0$ then by construction $\tau \wedge d\tau \in \Omega^{3,0}$. We now take ζ to be coordinates on \mathbb{CP}^1 , $\partial_{\bar{\zeta}}$ is the anti-holomorphic vertical tangent vector. It follows that $\partial_{\bar{\zeta}} \lrcorner d\tau \propto \tau$. Using coordinates adapted to the trivialisation we can write $\tau = \lambda (d\zeta + A_\mu dx^\mu)$ and $\tau \wedge \bar{\zeta} \lrcorner d\tau = 0$ implies $\partial_{\bar{\zeta}} A_\mu = 0$. A_μ being $\mathcal{O}(2)$ valued, it implies $A = A(x)^{A'}_{B'} \pi^{B'} \pi_{A'}$. Now 2.3 implies that $A^{A'}_{B'}$ is perfect.

$(iii) \Rightarrow (i)$

The connection A being perfect, we have from Proposition 2.3 that $\tau \wedge d\tau \wedge d\tau = 0$. From the definition of the almost complex structure, this implies $\tau \wedge d\tau = \tau \wedge F \pi \pi \in \Omega^{(3,0)}$. The connection being perfect, $F^i = \sigma^{\frac{\Lambda}{3}} \Sigma^i$ and the almost complex structure now coincide with the one described in proposition 2.2. The perfectness of the connection imply its integrability by proposition 2.3. \square

This should have made clear that this "non-linear graviton" theorem, with central equation $\tau \wedge d\tau \wedge d\tau = 0$, can be understood as a deep generalisation of the description of self-dual gravity in terms of perfect connection $F^i \wedge F^j = \frac{\delta^{ij}}{3} F^k \wedge F^k$, cf (1.2). As we already reviewed, full gravity can be described in terms of $SU(2)$ -connection only (cf Prop (1.3)) and this is therefore suggestive of a twistor description of full gravity in terms of the 1-form τ only.

3 Discussion on the would be "Twistor action for GR"

In [17] two new variational principles for Yang-Mills theory and conformal gravity based on fields living on twistor space were presented. The fact that the fields which appear in this action live on a 6d manifold ("projective twistor space") is compensated by new symmetries of the action and the propagating degrees of freedom thus remain the same as in the usual Yang-Mills or Conformal gravity action as expected. One of the nice features of these actions is that they give a natural explanation for why there is a MHV formalism for Yang-Mills and Conformal gravity. Because of the extra symmetries that these action enjoys (as compared to the space-time action) they allow to choose a special gauge (referred to as CSW gauge) that makes a MHV formalism manifest (cf [27], [22], [28] and [19]). If such an action existed for GR one could expect the same phenomenon and it could serve as a proof for the existence (or the obstruction to the existence) of a MHV formalism for GR.

As already described in section 2.2 a twistor action for self-dual gravity was presented in [21]. Then, in [19] a conjectured twistor action for full gravity was proposed. However, in spite of the many interesting features of this conjectured twistor action, some geometrical understanding is lacking, mainly because it is formulated around a fixed background, and this makes it unclear whether it actually describe GR or not.

Both the Twistor action for Yang-Mills and Conformal gravity were obtained by a generalisation of the respective space-time action. We very briefly sketch how this works for Yang-Mills, but refer the reader to [17] for details on the construction. This is of interest for us because we will see that, together with the description of metric in terms of connection described in section 1 it has an immediate generalisation to gravity.

3.1 The Twistor action for Yang-Mills from the Chalmer-Siegel action

In [17], the Chalmer-Siegel [29] action for Yang-Mills was taken as a starting point on the way to a twistor action:

$$S[A, B] = \int_M \text{Tr} \left(B \wedge F - \frac{\epsilon}{2} B \wedge B \right) \quad (27)$$

where B is taken to be a Lie algebra valued *self-dual* 2-form, ie $B = B_{A'B'} \Sigma^{A'B'}$. Here $\Sigma^{A'B'}$ is a basis of self-dual 2-forms associated with a fixed background flat metric and constructed as in (5). As already described, the Euclidean twistor space is the total space of the primed spinor bundle over M , the almost complex structure on \mathbb{PT} is given by taking the $(3, 0)$ -form on \mathbb{PT} to be $\Omega = \tau \wedge \Sigma^{A'B'} \pi_{A'} \pi_{B'} = \pi_{C'} d\pi^{C'} \wedge \Sigma^{A'B'} \pi_{A'} \pi_{B'}$.

An interesting feature of this action is that for $\epsilon = 0$ we are left with an action for self-dual Yang-Mills. This is a key point to make contact with twistor theory as self-dual Yang-Mills solutions are fully understood in terms of geometry of the twistor space through the Ward transform [30].

If we take the Penrose transform of $B_{A'B'}$ to be

$$B_{A'B'} = \int_{\mathbb{CP}^1} \pi_{A'} \pi_{B'} b \wedge \tau$$

(where b is a $(0, 1)$ -form on \mathbb{PT} with values in $\mathcal{O}(-4)$) and plug it into the action, we see that it is suggestive of the twistor action for Yang-Mills [17]:

$$S[a, b] = \int_{\mathbb{PT}} \text{Tr} (b \wedge f \wedge \Omega) - \frac{\epsilon}{2} \int_{M \times \mathbb{CP}^1 \times \mathbb{CP}^1} \text{Tr} \left(b_1 \wedge \tau_1 \wedge b_2 \wedge \tau_2 \left(\pi_{1A'} \pi_{2A'} \right)^2 \right). \quad (28)$$

Where now a is taken to be a $(0, 1)$ $SU(N)$ -connection of a Yang-Mills bundle *over* \mathbb{PT} and $f \in \Omega^{0,2}(\mathbb{PT})$ its curvature. For $\epsilon = 0$ varying the action with respect to b gives $f = 0$ and thus gives this bundle the structure of a holomorphic bundle over \mathbb{PT} . By a theorem from Ward [30], this is equivalent to self-dual Yang-Mills equations see also [20] for Euclidean methods in twistor theory. What's more, it turns out that this action describes full Yang-Mills for $\epsilon \neq 0$. Again, the aim here is just to sketch how this action is constructed and refer to [17] for a proper discussion.

3.2 A first twistor ansatz... and why it fails

We here would like to take as starting point the following space-time action:

$$S[A, \Psi] = \int \Psi^{ij} F^i \wedge F^j + \sum_{k=0}^{\infty} \left(\frac{3}{\Lambda} \right)^{k+1} (\psi^{k+2})^{ij} F^i \wedge F^j \quad (29)$$

This action is in fact an action for gravity. We explain in Appendix A how it can be derived by integrating fields from the Plebanski action.

This action, which does not seem to have attracted much attention up to now, has the following interesting interpretation: in the limit where Λ goes to infinity we recover an action for anti-self-dual gravity. For a finite Λ however this action describe full GR as an interacting theory around the anti-self-dual background with the cosmological constant playing the role of coupling constant. This parallels the Chalmers-Siegel action for Yang-Mills.

It suggests the following twistor ansatz,

$$S[\tau, \psi] = \int_{\mathcal{PT}} \psi \wedge \tau \wedge d\tau \wedge d\tau \quad (30)$$

$$+ \sum_{k=0}^{\infty} \left(\frac{3}{\Lambda} \right)^{k+1} \int_{M \times \mathbb{CP}^1 \times \mathbb{CP}^1} \psi_1 \wedge \tau_1 \wedge d\tau_1 \wedge \psi_2 \wedge \tau_2 \wedge d\tau_2 (\Psi)^k A^{B'}_{C'D'} \pi_1^{C'} \pi_1^{D'} \pi_{2A'} \pi_{2B'}$$

where τ is a $\mathcal{O}(2)$ -valued 1-form on \mathcal{PT} . As we already described in 2.3 such a 1-form is enough to construct an almost complex structure \mathcal{J}_τ on \mathcal{PT} and to give it a fibre bundle structure over some space-time M , $\mathbb{CP}^1 \hookrightarrow \mathcal{PT} \rightarrow M$.

This action also contains $\psi \in \Omega_{\mathbb{C}}^1 \otimes \mathcal{O}(-6)$ a 1-form on \mathcal{PT} with values in $\mathcal{O}(-6)$, its Penrose transform is $\Psi(x)^{A'B'}_{C'D'}$:

$$\Psi(x)^{A'B'}_{C'D'} = \int_{\mathbb{CP}^1_x} \psi \wedge \tau \pi^{A'} \pi^{B'} \pi_{C'} \pi_{D'}$$

where $\mathbb{CP}^1_x = P^{-1}(x)$ is the fibre above $x \in M$.

Interestingly, when Λ goes to infinity one recovers the Mason-Wolf action for self-dual gravity described in section 2.2. What's more, truncating the infinite sum to the first term one recovers an action that looks like a background independent version of the twistor action conjecture in [19].

Unfortunately, despite those encouraging features, one cannot prove that this twistor action is related to gravity. To do so one would hope that varying this action with respect to ψ would give enough field equations to recover a $SU(2)$ -connection from τ as was the case in our proof of the non-linear graviton theorem 2.3. However this does not seems to be the case here. We are thus unable to make contact with a space-time counter part of this action and the interpretation of the fields equations remains obscure.

3.3 A new action for Gravity as a background invariant generalisation of the Chalmers-Siegel action.

Let's now come back to the Chalmer-Siegel action and consider the special case of a $SU(2)$ -connection:

$$S[A, B] = \int_M B_{A'B'}^i \Sigma^{A'B'} \wedge F^i - \frac{\epsilon}{2} B_{A'B'}^i B_{C'D'}^i \Sigma^{A'B'} \wedge \Sigma^{C'D'} \quad (31)$$

Here $\Sigma^{A'B'}$ is a basis of self-dual 2-forms associated with a fixed background flat metric and constructed as in (5). This action has obviously nothing to do with an action for gravity as a background metric is present.

However, as explained in section 1, a definite $SU(2)$ -connection is enough to define a conformal class of metric. If we now choose a representative in the conformal class, and use it to parametrise the Σ 's, $\Sigma_A = \Sigma(g_A) = \Sigma(A)$, the action (31) becomes background independent:

$$S[A^i, B^{ij}] = \int B^{ij} \Sigma_A^i \wedge F^j - \frac{\epsilon}{2} B^{ij} B_{ik} \Sigma_A^j \wedge \Sigma_A^k$$

If we take B^{ij} to be unconstrained, the action ends up to be topological. However if we take B^{ij} to be traceless with the good choice of volume form then the action happens to describe gravity in the pure connection formulation:

Proposition 3.1.

The action

$$S[A^i, B^{ij}] = \int B^{ij} \Sigma_A^i \wedge F^j - \frac{\epsilon}{2} B^{ij} B_{ik} \Sigma_A^j \wedge \Sigma_A^k$$

, with Σ_A^i the basis of orthonormal self-dual 2-form associated with Urbantke metric with volume $\frac{1}{2\Lambda^2} \left(\text{tr} \sqrt{F \wedge F} \right)^2$ and B^{ij} a traceless matrix, describes the vacuum solution of Einstein equations with non zero cosmological constant. What's more for $\epsilon = 0$ this action describes anti-self-dual gravity.

Proof. By construction the Σ_A 's are such that,

$$F^i = M^{ij} \Sigma_A^j.$$

Our choice of volume form,

$$\frac{1}{3} \Sigma^i \wedge \Sigma^i = \frac{1}{2\Lambda^2} \left(\text{tr} \sqrt{F \wedge F} \right)^2,$$

is such that $\text{Tr} M = \Lambda$ is a constant.

Now, varying the action with respect to B , we get

$$M^{ij} \Big|_{\text{trace-free}} = \epsilon B^{ij}$$

which is equivalent to

$$F^i = \left(\epsilon B^{ij} + \frac{\Lambda}{3} \delta^{ij} \right) \Sigma^j.$$

For $\epsilon = 0$, these are the equations for self-dual gravity in terms of connection. For $\epsilon \neq 0$ we can solve for B , plugging this back into the action we obtain

$$S[A] = \frac{1}{\epsilon} \int \frac{1}{2} F^i \wedge F^i - \frac{1}{6} \left(\text{tr} \sqrt{F \wedge F} \right)^2.$$

Up to a topological term, this is just the pure connection action for gravity [3], see also Appendix A. \square

3.4 Discussion on a second ansatz

The action in Proposition (3.1) looks like a promising starting point to construct ansatz for twistor action for gravity. It indeed has many appealing features. First it explicitly separates the self-dual sector ($\epsilon = 0$) of the theory from the full theory ($\epsilon \neq 0$). Second it superficially looks like the space-time counterpart of the twistor action conjectured in [19]. Finally, as explained in the previous section the $SU(2)$ -connection on M can naturally be lifted as a 1-form τ on \mathcal{PT} . Starting with an action of this type would again allow to use the machinery described in section 2.

However as to now, despite many attempts from the author of this paper, none of the ansatz that are suggested by this action seem to lead to an interesting gravity action. We describe here one attempt that seemed at some point the most promising to the author. We will see that it can indeed eventually lead to a certain variational principle in twistor space but at the expense both of technical complications and the addition of an unnatural constraint. Thus the result seems both too complicated to be directly useful (let say for computing scattering amplitudes) and to anaesthetic to be otherwise appealing. However, on the way the interested reader should get some glimpses on the type of difficulties that one faces when one tries to construct such a variational principle in twistor space.

The essential idea here is it that we now would like to construct an action of the type $S[\tau, \psi]$ on \mathcal{PT} , some real 6d manifold. First using the results from section 2, one can construct an almost complex structure \mathcal{J}_τ which in turn allows to construct some space time M , giving \mathcal{PT} a fibre bundle structure $\mathbb{CP}^1 \hookrightarrow \mathcal{PT} \xrightarrow{\pi} M$. A look at the action 3.1 then suggests the following “twistor ansatz”:

$$S[\tau, \psi] = \int_{\mathbb{PT}} \psi \wedge \tau \wedge d\tau \wedge \Sigma_\tau + \frac{\epsilon}{2} B^{A'B'} \wedge B_{A'B'} \quad (32)$$

where Σ_τ should be constructed from τ only,

$$\psi \in \Lambda_{0,1} \otimes \mathcal{O}(-6)$$

and

$$B(x)^{A'B'} = \int_{\mathbb{CP}^1_x} \pi^{A'} \pi^{B'} \psi \wedge \tau \wedge \Sigma_\tau.$$

Where in this last line one should integrate over $\pi^{-1}(x) \simeq \mathbb{CP}^1$.

An appealing feature of actions of this type is that, linearising around a given background (let say describing flat space-time) we obtain $\delta\psi \in H^{0,1}(\mathcal{PT}, \mathcal{O}(-6))$ and $\delta\tau \in H^{0,1}(\mathcal{PT}, \mathcal{O}(2))$ which are then naturally interpreted as the Penrose transform of a propagating self-dual $\Psi_{A'B'C'D'}$ and anti-self-dual Ψ_{ABCD} gravitons.

The difficult part now is to make sense of Σ_τ . We propose the following. In section 2 we partly defined a connection a on $\mathcal{O}(2)$, through the relation

$$\bar{\tau} \wedge (d\bar{\tau} + d_a\tau)^2 = 0$$

however as for now it is only defined up to multiple of $\tau, \bar{\tau}$. If we require as some non-degeneracy condition that τ does not vanish on the \mathbb{CP}^1 that fibers \mathcal{PT} , we can then completely fix a by requiring that the $\mathcal{O}(2)$ -connection that induces on each \mathbb{CP}^1 fibres is the Levi-Cevita connection of the Kahler metric on \mathbb{CP}^1 . Now that a is completely defined we have access to its curvature $(d_a)^2$.

Consider the following triple of 2-forms: $(d_a\tau, d_a\bar{\tau}, (d_a)^2)$. Generically it spans a 3d subspace of the two-forms of each horizontal space and thus allows us to define a conformal metric (the associated Urbantke metric cf section 1) *on each horizontal tangent spaces*. Let's see how it works explicitly:

Defines

$$B^{A'B'} := \frac{\pi^{A'}\pi^{B'}}{(\pi.\hat{\pi})^2} d_a\bar{\tau} + \frac{\hat{\pi}^{A'}\hat{\pi}^{B'}}{(\pi.\hat{\pi})^2} d_a\tau + \frac{\pi^{(A'}\hat{\pi}^{B')}}{(\pi.\hat{\pi})^2} (d_a)^2$$

and

$$B^i = \sigma_{A'B'}^i B^{A'B'}.$$

This last object should be understood as an $\mathfrak{su}(2)$ -valued 2-form. From this we can follow the same procedure as in the first section and construct Σ :

$$\Sigma^i(x, \zeta) = X^{-\frac{1}{2}ij} B^j \quad \Sigma^{A'B'} = \sigma_i^{A'B'} \Sigma^i$$

such that $\Sigma^i \wedge \Sigma^j \propto \delta^{ij}$. It is associated with a conformal class of metric on each horizontal space $e^{AA'}(x, \pi)$, $\Sigma^{A'B'} = e^{A'}_A \wedge e^{B'A}$.

Importantly at this point the tetrad on the horizontal tangent space $e^{AA'}$ varies along the fiber $e^{AA'} = e^{AA'}(x, \pi)$.

In the end we define the $\mathcal{O}(2)$ -valued 2-form on \mathbb{PT} :

$$\Sigma_\tau(x, \pi) = \Sigma^{A'B'}(x, \pi) \pi_{A'} \pi_{B'} = \theta^0 \wedge \theta^1, \quad \text{with } \theta^A = e^{AA'} \pi_{A'}$$

This construction is not as arbitrary as it might seem at first sight: in the particular case where there is an underlying $SU(2)$ -connection such that $\tau = \pi_{A'} (d\pi^{A'} + A^{A'}_{B'} \pi^{B'})$, it precisely coincides with the construction from proposition 2.2. The connection a on $\mathcal{O}(2)$ then coincides with the connection described at the beginning of section 2 and the restriction of the triplet

$$(d_a\tau, d_a\bar{\tau}, (d_a)^2)$$

to the horizontal tangent space then indeed is just

$$\left(F^{A'B'} \pi_{A'} \pi_{B'}, F^{A'B'} \hat{\pi}_{A'} \hat{\pi}_{B'}, F^{A'B'} \pi_{A'} \hat{\pi}_{B'} \right).$$

Note that we did not need to assume the connection to be perfect. It can then be checked that, under such conditions, the twistor action (32) coincides with the original space-time action from proposition (3.1).

Therefore we could hope that with this definition for Σ_τ , the action (32) would describe gravity: all we need are the field equations for ψ to imply the existence of an $SU(2)$ connection such that $\tau = \pi_{A'} (d\pi^{A'} + A^{A'}_{B'} \pi^{B'})$.

At this point however, it seems that we are out of luck. Varying (32) with respect to ψ we obtain

$$\tau \wedge d\tau \wedge \Sigma_\tau + \epsilon \tau \wedge \Sigma_\tau \wedge B^{A'B'} \pi_{A'} \pi_{B'} = 0.$$

For simplicity let's consider the case $\epsilon = 0$. Then the action in proposition 3.1 is an action for self-dual gravity and we thus would like to interpret,

$$\tau \wedge d\tau \wedge \Sigma_\tau = 0 \tag{33}$$

as implying the integrability of some almost complex structure and/or as the perfectness of some $SU(2)$ connection arising on the way.

However, on the one hand, due to the important non linearities involved in constructing Σ_τ , it seems very complicated to interpret this field equations in terms of the almost complex structure from section 2.3. On the other hand, one could be tempted to consider as another almost complex structure: the one that makes $\tau \wedge \Sigma_\tau$ a $(3, 0)$ -form, then the field equations (33) just read $d\tau|_{0,2} = 0$.

At this point, to obtain self-dual gravity, it would be enough to be able to conclude that there exists a $SU(2)$ connection such that $\tau = \pi_{A'} (d\pi^{A'} + A^{A'}_{B'} \pi^{B'})$.

This is however not the case: generically τ can be written $\tau = d\zeta + A_\mu dx^\mu$ with $A_\mu \in \Gamma(\mathcal{O}(2))$. From this it follows that

$$\bar{\partial}_{\bar{\zeta}} \lrcorner d\tau = 0 \quad \Leftrightarrow \quad \bar{\partial}_{\bar{\zeta}} (A_\mu) dx^\mu = 0$$

would indeed imply that $A_\mu = A^{A'B'} \pi_{A'} \pi_{B'}$. On the other hand

$$\bar{\partial}_{\bar{\zeta}} \lrcorner (d\tau|_{0,2}) = 0 \quad \Leftrightarrow \quad \bar{\partial}_{\bar{\zeta}} (A_\mu) dx^\mu|_{0,1} = 0$$

are just not enough field equations to conclude that $\bar{\partial}_{\bar{\zeta}} (A_\mu) = 0$. In this last case one indeed misses one half of the necessary field equations:

$$\bar{\partial}_{\bar{\zeta}} \lrcorner (d\tau|_{1,1}) = 0.$$

In principle, this missing set of equations could be implemented as a constraint in our twistor ansatz (32): this would at last give a twistor action for gravity. However, on top of definitely spoiling any remaining geometric aesthetics, it would also add another layer of complexity to our already complicated construction, making it more than unlikely to be useful.

Conclusion

In this paper we reviewed chiral formulations of gravity, in particular the pure connection formulation, and showed that they nicely interact with twistor theory: from a definite $SU(2)$ -connection only on a 4d manifold M one can construct an almost hermitian structure on the associated twistor space $\mathbb{PT}(M)$, cf proposition 2.2. The holomorphicity of a certain

1-form τ is then equivalent to the requirement that the connection is the self-dual part of an anti-self-dual space-time, cf proposition 2.3. On the other hand such a 1-form τ on a 6d manifold \mathcal{PT} defines an almost complex structure \mathcal{J}_τ and a 4d manifold $\mathbb{CP}^1 \hookrightarrow \mathcal{PT} \rightarrow M$. Finally, integrability of this almost complex structure allows to define a $SU(2)$ -connection such that it is the self-dual connection of an anti-self-dual metric, cf proposition 2.5.

Chiral formulations of gravity come in the form of many different Lagrangians that we reviewed in Appendix A and together with our description of twistor theory in terms of connections they suggest different ansatz for a twistor action for gravity. We described some of them in section 3. As to now however, none of them seems to lead to a useful twistor action and the chase for such an action (if only it exists!) is still open.

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A A Review of Chiral Lagrangians for Gravity

A.1 The Plebanski action, $S[A, \Psi, B]$:

The Plebanski Action for General Relativity is

$$S[A, B, \Psi] = \int B^i \wedge F^i + \left(\Psi^{ij} + \frac{\Lambda}{3} \delta^{ij} \right) B^i \wedge B^j \quad (34)$$

see [1], [5] for the original references.

It is not a very economical action as it contains many fields: a $Su(2)$ -connection A (which does not need to be a definite connection at this stage), a $su(2)$ -valued 2-form B (again we do not need to require this triplet of 2-forms to be a definite triplet) and a symmetric traceless field: $\psi^{(ij)} = \psi^{ij}$.

Varying the action with respect to Ψ^{ij} we get

$$\frac{\delta S}{\delta \Psi^{ij}} = 0 \quad \Leftrightarrow \quad B^i \wedge B^j \propto \delta^{ij} d^4x. \quad (35)$$

Thus the triplet of 2-form $\{B^i\}_{i \in 1..3}$ is definite: It implies that there is a unique Euclidean, invertible, conformal metric \tilde{g}_B such that the triplet is a basis of self-dual 2-forms. What's more the field equations implies that $\{B^i\}_{i \in 1,2,3}$ is an orthonormal basis for the wedge product. This means that, up to a $SU(2)$ action, the B 's can in fact be constructed from the conformal metric \tilde{g}_B as in (5). We can pick up a representative g_B in the conformal class \tilde{g}_B by requiring the volume form to be $vol_g = B^i \wedge B^i$.

Varying the action with respect to A we then have

$$\frac{\delta S}{\delta A^i} = 0 \quad \Leftrightarrow \quad d_A B^i = 0. \quad (36)$$

This equation, together with (35), implies that A is the self-dual part of the Levi-Cevita connection associated with the metric g_B .

Varying the action with respect to B we find,

$$\frac{\delta S}{\delta B^i} = 0 \quad \Leftrightarrow \quad F^i = \left(\Psi^{ij} + \frac{\Lambda}{2} \delta^{ij} \right) B^j. \quad (37)$$

which is now just the Chiral version of Einstein equations, cf (8).

Note that this action allows simple modifications leading to theories dubbed “neighbours of GR” by allowing Λ to be a function of Ψ (this class of modified theories was described in [31], [32]):

$$S[A, B, \Psi] = \int B^i \wedge F^i + \left(\Psi^{ij} + \frac{\Lambda(\Psi)}{3} \delta^{ij} \right) B^i \wedge B^j \quad (38)$$

All those theories somewhat surprisingly have the same number of propagating degrees of freedom as GR.

A.2 The self-dual Palatini action $S[A, e]$

The self-dual Palatini action (or Ashtekar action) really is the covariant side of the canonical description of gravity in terms of Ashtekar variables (see [2], [33] for a precise derivation of the constraints). One can obtain this action by varying ψ in the Plebanski action and solving the associated “simplicity constraints” $B^i \wedge B^j \propto \delta^{ij}$. As we already stated this can be done by introducing the unique tetrad $e^I_{I \in 0..3}$ such that

$$\left\{ B^i = \Sigma^i_{IJ} \frac{e^I \wedge e^J}{2} = -e^0 \wedge e^i - \frac{\epsilon^{ijk}}{2} e^j \wedge e^k \right\}_{i \in 1,2,3}.$$

$$S[A, e] = \int B^i(e) \wedge F^i + \Lambda \frac{\epsilon_{IJKL}}{4} e^I \wedge e^J \wedge e^K \wedge e^L \quad (39)$$

Varying the action with respect to the connection we get the compatibility equation $d_A B^i = 0$. Using the decomposition (52), one sees that integrating out A from the self-dual Palatini action gives the usual Einstein Hilbert.

A.3 Intermediate actions of the type $S[A, \Psi]$:

Let’s start again from the Plebanski action, now instead of integrating out ψ , we want to eliminate the B field: Using equations (37), we obtain the following action:

$$S[A, \psi] = \int \left(\left(\Psi + \frac{\Lambda}{3} \delta \right)^{-1} \right)^{ij} F^i \wedge F^j \quad (40)$$

We let to the reader to check that the this action yields field equations for GR. As for all chiral formulations of GR the mechanism comes done to constructing a metric with Urbantke trick (4), and the checking the compatibility relations (6) $d_A \Sigma = 0$ (here $\Sigma^i = \left(\left(\Psi + \frac{\Lambda}{3} \delta \right)^{-1} \right)^{ij} F^j$).

This action comes in a few different variants that each have their interests. A first variant with Lagrange multiplier that first appeared in [34] allows to see again GR as part of the larger class of "neighbourhood of GR" :

$$S[A, \psi] = \int \Psi^{ij} F^i \wedge F^j + \mu f(\Psi) \quad (41)$$

Each constraint $f(\Psi)$ will give a different theory. One can recover GR by considering $f(\Psi) = \text{Tr}(\Psi^{-1}) - \Lambda$. Interestingly one can easily describe "anti-self-dual gravity" in this formulation, it is associated with the constraint $f(\Psi) = \text{Tr}\Psi$. With this constraint, fields equations indeed reads,

$$F^i \wedge F^j = \frac{\delta^{ij}}{3} F^k \wedge F^k. \quad (42)$$

A second variant of the action (40) is obtained by expanding the inverse matrix in power of Ψ ,

$$S[A, \Psi] = \frac{3}{\Lambda} \int F^i \wedge F^i + \sum_{k=1}^{\infty} \left(\frac{3}{\Lambda} \right)^k (\psi^k)^{ij} F^i \wedge F^j \quad (43)$$

By subtracting the topological term $F^i \wedge F^i$ and multiplying by $\left(\frac{\Lambda}{3}\right)^2$ we obtain:

$$S[A, \Psi] = \int \Psi^{ij} F^i \wedge F^j + \sum_{k=0}^{\infty} \left(\frac{3}{\Lambda} \right)^{k+1} (\psi^{k+2})^{ij} F^i \wedge F^j \quad (44)$$

This action, which does not seem to have attracted much attention up to now, has the following interesting interpretation: in the limit where Λ goes to infinity we recover an action for anti-self-dual gravity. For a finite Λ however this action describe full GR as an interacting theory around the anti-self-dual background with the cosmological constant playing the role of coupling constant. This parallels the Chalmers-Siegel action for Yang-Mills.

A.4 Intermediate actions of the type $S[A, B]$

Before we come to the pure connection action let's consider the action for gravity of the form $S[A, B]$. It has already been described in [35] and cannot be obtained from the Plebanski action (see however [36] for a derivation from a more complicated Lagrangian):

$$S[A, B] = \int B^i \wedge F^i + \left(\text{Tr} \sqrt{B \wedge B} \right)^2 + \frac{\epsilon}{2} B^i \wedge B^i$$

As described in [35], for $\epsilon = 0$ one recovers again anti-self-dual gravity while for $\epsilon \neq 0$ it describes full gravity.

This Lagrangian can also be generalised (cf [37]) to describe "neighbours of GR" :

$$S[A, B] = \int B^i \wedge F^i + V(B \wedge B)$$

Where V is a potential that parametrize the class of modified theories.

A.5 Pure connection action $S[A]$:

We now come to the pure connection action, starting again from (40) and integrating Ψ^{ij} one obtains:

$$S[A] = \int \left(\text{Tr} \sqrt{F \wedge F} \right)^2. \quad (45)$$

See Prop 1.3 for a discussion on the pure connection Einstein field equations and [3] for the original reference.

This can again be easily generalised to describe all the “neighbours of GR”:

$$S[A] = \int f [F \wedge F].$$

Where f is a function that parametrizes the class of modified theories (see [38] for more details).

A.6 The background independent Chalmer-Siegel action for $SU(2)$

We end up this review of chiral Lagrangians with the action described in proposition 3.1 and that was previously unknown in the literature:

As we already explained in section 3.1, using the property of definite $SU(2)$ -connections to parametrize space-time metric, one can turn the Chalmer-Siegel action for $SU(2)$ connection:

$$S[A, B] = \int_M \text{Tr} \left(B \wedge F - \frac{\epsilon}{2} B \wedge B \right)$$

into a background independent action:

$$S[A^i, B^{ij}] = \int B^{ij} \Sigma_A^i \wedge F^j - \frac{\epsilon}{2} B^{ij} B_{ik} \Sigma_A^j \wedge \Sigma_A^k.$$

As described in proposition 3.1, for $\epsilon \neq 0$ this action describes gravity at the condition of B^{ij} to be traceless. In the case where $\epsilon = 0$ this action describes anti-self-dual gravity.

B Decomposition of the Curvature in coordinates

In this appendix we prove, using coordinates, the different claims made in the first part of section 2.

In this appendix we use freely the isomorphism $\mathfrak{so}(4) \simeq \Lambda^2$ to represent elements of $\mathfrak{so}(4, \mathbb{R})$ as 2-forms. I.e, we pick up a basis of 1-forms, $\{e^I\}_{I \in \{1 \dots 4\}}$ compatible with the metric, $ds^2 = e^I \otimes e^I$, and write for $\mathbf{b} \in \mathfrak{su}(2)$ as $\mathbf{b} = b_{IJ} \frac{e^I \wedge e^J}{2}$, with abuse of notation. The metric allows to raise and lower $I, J, K \dots$ indices. With this notations, the Lie bracket reads,

$$\mathbf{a}, \mathbf{b} \in \mathfrak{so}(4), \quad [\mathbf{a}, \mathbf{b}] = (a_I^K b_{KJ} - b_I^K a_{KJ}) \frac{e^I \wedge e^J}{2}.$$

Then for any $\mathbf{b} \in \mathfrak{su}(2)$ the decomposition $\mathfrak{so}(4) = \mathfrak{su}(2) \otimes \mathfrak{su}(2)$ reads:

$$b_{IJ} \frac{e^I \wedge e^J}{2} = B^i \frac{\Sigma^i}{2} + \tilde{B}^i \frac{\tilde{\Sigma}^i}{2}. \quad (46)$$

Where the Sigma tensors coincide with the one described in section 2. In terms of the tetrad they take the explicit form:

$$\left\{ \Sigma^i = -e^0 \wedge e^i - \frac{\epsilon^{ijk}}{2} e^j \wedge e^k \right\}_{i \in 1,2,3}, \quad \left\{ \tilde{\Sigma}^i = e^0 \wedge e^i - \frac{\epsilon^{ijk}}{2} e^j \wedge e^k \right\}_{i \in 1,2,3}. \quad (47)$$

They form a basis of self-dual and anti-self-dual 2-forms respectively. It is orthogonal for the wedge product:

$$\Sigma^i \wedge \Sigma^j = \tilde{\Sigma}^i \wedge \tilde{\Sigma}^j = 2\delta^{ij} e^0 \wedge e^1 \wedge e^2 \wedge e^3, \quad \Sigma^i \wedge \tilde{\Sigma}^j = 0.$$

As was already stated in the main part of this paper, the decomposition of Lie algebra $\mathfrak{so}(4) = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ corresponds to the decomposition $\Lambda^2 = \Lambda_+^2 \oplus \Lambda_-^2$ of 2-forms:

$$[\Sigma^i, \Sigma^j] = 2\epsilon^{ijk} \Sigma^k, \quad [\tilde{\Sigma}^i, \tilde{\Sigma}^j] = 2\epsilon^{ijk} \tilde{\Sigma}^k, \quad [\Sigma^i, \tilde{\Sigma}^j] = 0.$$

In what follows we will make an important use of the tensors, $\Sigma_{IJ}^i, \tilde{\Sigma}_{IJ}^i$ defined by $\Sigma^i = \Sigma_{IJ}^i \frac{e^I \wedge e^J}{2}, \tilde{\Sigma}^i = \tilde{\Sigma}_{IJ}^i \frac{e^I \wedge e^J}{2}$. They verify the algebra,

$$\Sigma_{IK}^i \Sigma_{JK}^j = -\delta^{ij} g_{IJ} + \epsilon^{ijk} \Sigma_{IJ}^k, \quad \tilde{\Sigma}_{IK}^i \tilde{\Sigma}_{JK}^j = -\delta^{ij} g_{IJ} + \epsilon^{ijk} \tilde{\Sigma}_{IJ}^k, \quad \Sigma_{IK}^i \tilde{\Sigma}_{JK}^j = \sigma_{IJ}^{ij}. \quad (48)$$

Where σ_{IJ}^{ij} is some symmetric traceless tensor, $\sigma_{IJ}^{[ij]} = 0, \sigma_{IJ}^{kk} = 0$.

Decomposition of the Curvature tensor in coordinates

Consider a 4d Riemannian manifold $\{g, M\}, \{e^I\}_{I \in 0..4}$ an orthonormal co-frame. The Levi-Cevita connection, ∇ , then naturally is a $SO(4)$ -connection. We will write its potential 1-form \mathbf{a} and curvature 2-form \mathbf{f} as

$$a^I{}_J = a^I{}_{JK} e^K, \quad f^I{}_J = da^I{}_J + a^I{}_K \wedge a^K{}_J = f^I{}_{JKL} \frac{e^K \wedge e^L}{2}.$$

Note that the Riemann curvature f here is a $\mathfrak{so}(4)$ -valued 2-form.

Now we can use the decomposition $\mathfrak{so}(4) = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$, concretely realised as (46), to define the chiral connections (D, \tilde{D}) with potential (A, \tilde{A}) as

$$a^I{}_J = A^i \frac{\Sigma^i I{}_J}{2} + \tilde{A}^i \frac{\tilde{\Sigma}^i I{}_J}{2}. \quad (49)$$

These connections naturally are $SU(2)$ -connections. In section 2 we stated that these connections are compatible with $\Sigma^i, \tilde{\Sigma}^i$ in the following sense:

$$d_A \Sigma^i = 0, \quad d_{\tilde{A}} \tilde{\Sigma}^i = 0. \quad (50)$$

We can prove this by a direct computation:

$$\begin{aligned}
d_A \Sigma^i &= \frac{1}{2} d_A (\Sigma_{IJ}^i e^I \wedge e^J) \\
&= \frac{1}{2} d_A (\Sigma_{IJ}^i) e^I \wedge e^J \\
&= \frac{e^I \wedge e^J}{2} \wedge (\epsilon^{ijk} A^j \Sigma_{IJ}^k - 2 \Sigma_{IK}^i a^K{}_J) \\
&= \frac{e^I \wedge e^J}{2} \wedge \left(\epsilon^{ijk} A^j \Sigma_{IJ}^k - 2 A^j \Sigma_{IK}^i \frac{\Sigma^{JK}{}_J}{2} - 2 \tilde{A}^j \Sigma_{IK}^i \frac{\tilde{\Sigma}^{JK}{}_J}{2} \right) \\
&= 0
\end{aligned}$$

where as step 1 we used the torsion freeness of a (ie $d_a e^I = 0$), step 2 is just the decomposition of the Levi Cevita connection into its chiral parts, (ie, eq(49)) and at step 3 we made use the algebra (48).

As already stated in the main body of this paper, the relations (50) can be used as an alternative way of defining A (resp \tilde{A}) as the unique $SU(2)$ -connection compatible with Σ^i (resp $\tilde{\Sigma}^i$).

In complete parallel with (49) we define the "self-dual part of the Curvature" F and the "anti-self-dual part of the Curvature" \tilde{F} as

$$f^I{}_J = F^i \frac{\Sigma^{iI}{}_J}{2} + \tilde{F}^i \frac{\tilde{\Sigma}^{iI}{}_J}{2}, \quad (51)$$

and these are naturally $\mathfrak{su}(2)$ -valued two-forms. In fact we have,

$$F^i = dA^i + \frac{\epsilon^{ijk}}{2} A^j \wedge A^k, \quad \tilde{F}^i = d\tilde{A}^i + \frac{\epsilon^{ijk}}{2} \tilde{A}^j \wedge \tilde{A}^k,$$

as can be seen using the algebra (48). Ie, the (anti-)self-dual part of the curvature is the curvature of the (anti-)self-dual connection.

Now F^i , \tilde{F}^i being ($\mathfrak{su}(2)$ -valued) 2-forms, we can decompose them into self-dual and anti-self-dual pieces:

$$F^i = F^{ij} \Sigma^j + G^{ij} \tilde{\Sigma}^j, \quad \tilde{F}^i = \tilde{G}^{ij} \Sigma^j + \tilde{F}^{ij} \tilde{\Sigma}^j.$$

This is just another way of writing the bloc decomposition (1.1). The Riemann curvature now reads

$$f^I{}_J = \frac{1}{2} \left(F^{ij} \Sigma^j \Sigma^{iI}{}_J + G^{ij} \Sigma^j \tilde{\Sigma}^{iI}{}_J + \tilde{G}^{ij} \tilde{\Sigma}^j \Sigma^{iI}{}_J + \tilde{F}^{ij} \tilde{\Sigma}^j \tilde{\Sigma}^{iI}{}_J \right).$$

Again, this is just another version of the bloc decomposition (1.1). To get the final form of the decomposition we write

$$F^{ij} = \frac{\lambda}{3} \delta^{ij} + \psi^{ij}, \quad \tilde{F}^{ij} = \frac{\tilde{\lambda}}{3} \delta^{ij} + \tilde{\psi}^{ij},$$

white ψ , $\tilde{\psi}$ some traceless tensors and $\lambda = \text{tr}F$, $\tilde{\lambda} = \text{tr}\tilde{F}$. Finally can write the following decomposition:

$$f^I{}_J = W^I{}_J + \tilde{W}^I{}_J + \frac{1}{2} \left(G^{ij} \Sigma^j \tilde{\Sigma}^i{}_J + \tilde{G}^{ij} \tilde{\Sigma}^j \Sigma^i{}_J \right) + \frac{\lambda}{3} \Sigma^i \frac{\Sigma^i{}_J}{2} + \frac{\tilde{\lambda}}{3} \tilde{\Sigma}^i \frac{\tilde{\Sigma}^i{}_J}{2} \quad (52)$$

where

$$\begin{aligned} W^I{}_J &= \psi^{ij} \Sigma^j \frac{\Sigma^i{}_J}{2} && \text{is the self-dual part of the Weyl tensor,} \\ \tilde{W}^I{}_J &= \tilde{\psi}^{ij} \tilde{\Sigma}^j \frac{\tilde{\Sigma}^i{}_J}{2} && \text{is the anti-self-dual part of the Weyl tensor,} \\ \frac{1}{2} G^{ij} \tilde{\Sigma}^i_{KI} \Sigma^{jK}{}_J + \frac{1}{2} \tilde{G}^{ij} \Sigma^i_{KI} \tilde{\Sigma}^{jK}{}_J &&& \text{is the traceless Ricci tensor,} \\ 2\lambda + 2\tilde{\lambda} &&& \text{is the Scalar curvature.} \end{aligned} \quad (53)$$

These can be related to the usual definitions by contracting indices in (52) and using the algebra (48):

$$\begin{aligned} \text{Scalar curvature:} \quad & f^{IJ}{}_{IJ} = 2\lambda + 2\tilde{\lambda} \\ \text{Traceless Ricci:} \quad & f_{K(I}{}^{K}{}_{J)} - \frac{1}{4} f^{KL}{}_{KL} g_{IJ} = \frac{1}{2} G^{ij} \tilde{\Sigma}^i_{KI} \Sigma^{jK}{}_J + \frac{1}{2} \tilde{G}^{ij} \Sigma^i_{KI} \tilde{\Sigma}^{jK}{}_J. \end{aligned}$$

As stated in the main body of this paper the torsion freeness of the Levi-Cevita connection implies that the Riemann tensor has some internal symmetries that lead to further simplifications:

$$\begin{aligned} f_{IJKL} = f_{KLIJ} &\Rightarrow \psi^{ij} = \psi^{(ij)}, \quad \tilde{\psi}^{ij} = \tilde{\psi}^{(ij)}, \quad \tilde{G}^{ij} = G^{ji}. \\ f_{I[JKL]} = 0 &\Leftrightarrow f_{NIKL} \epsilon^{NJKL} = 0 \Rightarrow \lambda = \tilde{\lambda}. \end{aligned}$$

The second relation follows from using the (anti)-self duality of the sigma tensors.

With those symmetries, Einstein equations

$$R_{IJ} = \Lambda g_{IJ},$$

are equivalent to

$$F^i = \left(\psi^{ij} + \frac{\Lambda}{3} \delta^{ij} \right) \Sigma^j \quad (54)$$

(ie $G = 0$, $\lambda = \Lambda$) and that we therefore only need one half of the Riemann tensor to state the

C Spinor conventions

We convert $\mathfrak{su}(2)$ lie algebra indices into spinor notations according to the rule:

$$V = V^i \tau_i \in \mathfrak{su}(2), \quad V^i = \{x, y, z\} \Leftrightarrow V^i \tau_i^{A'}{}_{B'} = V^{A'}{}_{B'} = \frac{1}{2i} \begin{pmatrix} z & x - iy \\ x + iy & -z \end{pmatrix} \in \mathfrak{su}(2).$$

Such that $[\tau^i, \tau^j] = \epsilon^{ijk} \tau^k$. Latin indices are raised and lowered with the flat metric δ^{ij} , spinor indices are raised and lowered as usual using the antisymmetric $\epsilon^{A'B'}$ as explained in the beginning of section 2. We go from one type of indices to the other as follows:

$$V^{A'B'} = V^i \tau_i^{A'B'} \quad \Leftrightarrow \quad V^i = 2\tau_{A'B'}^i V^{A'B'}.$$

On the other hand, to convert space-time indices into spinor ones we use the convention:

$$V^I e_I^{AA'} = \frac{1}{i\sqrt{2}} \begin{pmatrix} -it + z & x - iy \\ x + iy & -it - z \end{pmatrix}, \quad V^I = \{t, x, y, z\}.$$

With this conventions, a direct computation shows that

$$\Sigma^i = 2\tau_{A'B'}^i \frac{1}{2} e^{A'C} \wedge e^{B'}{}_C = -e^0 \wedge e^i - \epsilon^{ijk} e^i \wedge e^k, \quad i, j, k \in \{1, 2, 3\}.$$

ie

$$\Sigma^{A'B'} = \tau_i^{A'B'} \Sigma^i = \frac{1}{2} e^{A'C} \wedge e^{B'}{}_C.$$

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